A TREATISE

ON

PLANE AND SPHERICAL TRIGONOMETRY.

BY THE

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PREFACE.

Before a student can successfully attempt the perusal of works on Physical Science, he will find it expedient to have some acquaintance with the principles of Trigonometry. For a student deficient in this knowledge, the following Treatise was written; and its object is to teach so much of Trigonometry as may be necessary for his purpose, and no more.

Keeping this object in view, no fantastic combinations of chords, or secants, or versed sines, have been allowed to perplex the reader, or to withdraw him from the study of propositions which are all-important. No theorem has been introduced which is not either of itself eminently useful, or upon which important reasonings may not be founded.

The very copious Table of Contents exhibits a plan of the work; and it may be profitably used as a general Syllabus of Trigonometry.

The Introduction, and the first four Chapters, contain the principles of Plane Trigonometry; and every theorem in them must be carefully impressed upon the memory of the student, before he directs his attention to higher subjects.

The propositions in the fifth Chapter are curious, and deserving of attention, particularly that in which the area of the circle is identified with that of a polygon of an infinite number of sides.

The theorems demonstrated in the sixth Chapter are so important in the applications of Analysis to Physics, that a Treatise upon Trigonometry would be considered incomplete without them. But no theorem has been introduced, the proof of which demands more than a tolerably good acquaintance with the first part of Algebra, and the Binomial Theorem; and it is confidently hoped that what has been written will be easily understood by the careful reader.

In the Treatise on Spherical Trigonometry, the early Chapters which contain the formulas for the solution of right and oblique-angled spherical triangles, are the most important; and when the reader is familiarized with them, he may without difficulty understand the calculations of Practical Astronomy, a science to which the latter part of this work is almost wholly subservient.

The work is concluded by two Chapters containing formulas for finding the area of the spherical triangle, and various propositions of Solid Geometry, respecting the regular polyhedrons, and the parallelopipedon, without which a Treatise on Spherical Trigonometry would be considered incomplete.

T. G. HALL.

KING'S COLLEGE, LONDON,
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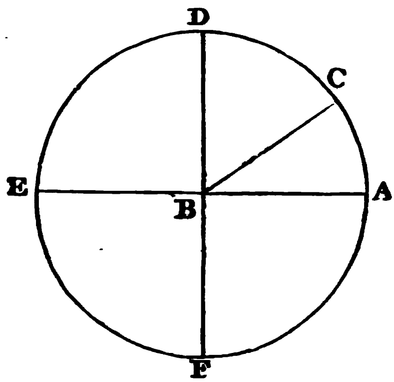
Now, the magnitude of AD depends both upon the length of AB, and the inclination of AB to BC; AD must therefore be found in terms of AB and the $\angle$ ABC, before the area of the triangle can be ascertained.

We shall hereafter see in what manner AD is connected with AB and the angle ABC.

Fig. 2



3. The arcs of the circle afford a means of connecting angular space with lines.

Let ABC be an angle; make $BC=BA$; with centre B, and radius BA, describe a circle. Produce AB to E, and draw $DBF \perp$ to ABE. The circle will be divided into four equal parts, called quadrants, and the angles, ABD, DBE, EBF, ABF, are right angles.

Now by Euclid, Book VI. Prop. 33.

$$\angle\, ABC : \angle\, ABD :: \text{arc } AC : \text{arc } AD;$$

$$\therefore \angle\, ABC : 4 \angle\, ABD :: \text{arc } AC : 4 \times \text{arc } AD.$$

But $4 \times \angle\, ABD$ is 4 right angles, which is a constant quantity, and $4 \times$ arc AD is the whole circumference, which, for the same circle, is constant.

Hence, $\angle ABC \propto$ arc AC, or the arc is the measure of the angle.

4. We must now seek for some method, by which an angle and an arc may be expressed numerically.

In general, a right angle is divided into 90 equal parts, called degrees; each of these degrees into 60 parts, called minutes; each minute into 60 parts, called seconds; each second into 60 parts, called thirds, and so on; and these are denoted by $^\circ\ '\ ''$, &c. Thus, 27 degrees, 35 minutes, 10 seconds, are written $27^\circ\ 35'\ 10''$.

5. The quadrant AD, which subtends the right angle ABD, is also called an arc of 90 degrees, for the quadrant may be divided into 90 equal parts, each of which will subtend a degree, the ninetieth part of the angle ABD. So also we talk of any other arc as being an arc of so many degrees.

6. The ratio of the circumference of a circle to its diameter is a constant ratio, very nearly equal to the number 3.14159265; this number is usually expressed by the Greek letter π; and thus, if C be the circumference, and r be the radius of a circle,

$$\frac{C}{2r} = \pi; \therefore C = 2\pi r.$$

Hence, the length of a semicircle is πr, and of a quadrant $\frac{\pi r}{2}$.

7. We have now another method of expressing the value of an angle.

Let an arc a subtend an angle of A°.

Then, $\because \frac{\pi r}{2}$ subtends 90°;

$$\therefore \frac{A^\circ}{90^\circ} = \frac{a}{\frac{\pi r}{2}} = \frac{2a}{\pi r};$$

$$\therefore A^\circ = \frac{180^\circ}{\pi} \cdot \frac{a}{r};$$

or the number of degrees in any angle is found by multiplying the fraction $\left(\frac{a}{r}\right)$ by the constant multiplier $\frac{180^\circ}{\pi}$.

8. This expression may again be modified; for let m° represent the degrees in an angle subtended by an arc equal to the radius;

$$\therefore m^\circ = \frac{180^\circ}{\pi} \cdot \frac{r}{r} = \frac{180^\circ}{\pi} = 57^\circ.29578;$$

$$\therefore A^\circ = m^\circ \left(\frac{a}{r}\right).$$

9. It is usual in the higher mathematics to express an angle by the fraction $\frac{a}{r}$ or $\frac{arc}{rad}$; whenever this is the case, it must be remembered, that $\frac{a}{r}$ is only a part of m°; or, that m is the unit of angular measure of which $\left(\frac{a}{r}\right)$ is the multiple or part. It may also be observed, that the fraction $\frac{a}{r}$ is called the circular measure of the angle.

COROLLARY. Hence, if the arc be a quadrant, or $a = \frac{\pi r}{2}$; $\frac{a}{r} = \frac{\pi}{2}$, or $\frac{\pi}{2}$ is equal to the circular measure of a quadrant; or we may express a right angle by the number $\frac{\pi}{2}$, and, $\therefore$ two right angles by π.

Example 1. The radius of a circle being 100 feet, find the length of an arc of 30°.

$$30° = \frac{180°}{\pi} \cdot \frac{a}{100}; \quad \therefore a = \frac{314.159}{6} = 52.359.$$

Example 2. The circular magnitude of an angle is $\frac{2}{5}$; find the degrees in it.

$$A° = \frac{2}{5} m° = \frac{4}{10}(57°.29578) = 22°.918312.$$

10. We may now obtain some useful results for the angles of equilateral and equiangular polygons.

Prop. Find the interior angle of a regular polygon of (n) sides.

Let A be the common angle; $\therefore nA =$ sum of all the angles.

Hence, by Euclid, Prop. 32, Cor. 1.

$$nA = 2n \cdot \frac{\pi}{2} - 4 \cdot \frac{\pi}{2} = (n-2)\pi.$$

$$\therefore A = \frac{n-2}{n}\pi = \pi - \frac{2\pi}{n};$$

or, expressed in degrees,

$$A° = 180° - \frac{360°}{n}.$$

Example. Find the interior angle of a regular decagon.

Here $n = 10$; $\therefore A° = 180° - 36° = 144°$.

Cor. The expression for the angle A, $A = \pi - \frac{2\pi}{n}$, shows that as the number of the sides increases, or as the fraction $\frac{2\pi}{n}$ decreases, the angle increases; but the

angle has a limit which it cannot exceed, for so long as the fraction $\frac{2\pi}{n}$ exists, the angle must be less than π.

11. The division of the right angle into 90 parts, and each of these into 60 parts, and so on, was universally adopted up to the end of the last century; when it was proposed by the French geometricians to divide the right angle into 100 parts, called grades, each grade into 100 parts, for minutes, and so on; by which means, the minutes, seconds, &c. being decimal fractions, of which the integer is the grade, all the operations of multiplication and division, &c. are at once performed, without further reduction.

The reduction of French grades to English degrees, or the converse, may be effected in a simple manner.

For if $\frac{\pi}{2}$ represent a right angle, $\frac{\pi}{180}$ and $\frac{\pi}{200}$ will express the magnitude of an English and French degree respectively.

Let E and F be the number of English and French degrees in the same angle; then, since the whole angle = magnitude of one degree × number of degrees in the angle,

$$\text{Angle} = E \times \frac{\pi}{180} = \text{also } F\frac{\pi}{200}.$$

$$\therefore \frac{E}{9} = \frac{F}{10}; \quad \therefore E = \frac{9F}{10} = F - \frac{F}{10}.$$

Whence this rule;—to convert French degrees into English, remove the decimal place one to the left, and subtract the new decimal from the original one; the integer part of the remainder will give the number of English degrees, and the decimal part, the minutes, seconds, &c.

EXAMPLE. Find the number of English degrees in 37° 5′ 28″ French.

$$
\begin{array}{rl}
37^\circ\ 5'\ 28'' = 37.0528 & = F \\
3.70528 & = \dfrac{F}{10} \\
\hline
33.34752 & \\
60 & \\
\hline
20.85120 & \\
60 & \\
\hline
51.07200 & \text{ANSWER, } 33^\circ\ 20'\ 51''\ .072.
\end{array}
$$

Again, since $E = \dfrac{9F}{10}$, we have $F = \dfrac{10E}{9} = E + \dfrac{E}{9}$, a formula by which English degrees are converted into French.

RULE. Having reduced the minutes and seconds to the decimal of a degree, add $\frac{1}{9}$th part of the whole, and the sum will give the French degrees, minutes, &c.

Find the French degrees in 33° 20′ 51″ .072 English.

$$
\begin{array}{rl}
60)\ 51.072 & \\
\hline
60)20,8512 & \\
\hline
33.34752 & = E \\
3.70528 & = \dfrac{E}{9} \\
\hline
37.05280 & \text{ANSWER, } 37^\circ\ 5'\ 28''.
\end{array}
$$

QUESTIONS.

Ex. 1. Express 90° 8′ 7″ French in the English scale.

Ex. 2. Express 81 4′ 21″ .468 English in the French scale.

Ex. 3. If F' and F'', E' and E'', represent the magnitude of a French and English minute and second respectively, show that,

$$\frac{F'}{E'} = \frac{3.3^2}{2.5^2}; \quad \frac{F''}{E''} = \frac{3.3^3}{2.5^3}.$$

Ex. 4. Compare the interior angles of a regular octagon and dodecagon.

Ex. 5. The earth being supposed a sphere, of which the diameter is 7980 miles, find the length of an arc of 1°.

Ex. 6. Find the diameter of a globe, when an arc of 25° of the meridian measures four feet.

Ex. 7. Find the number of degrees in a circular arc, of which the radius is 25 feet, the arc being 30 feet long.

Ex. 8. Find the number of degrees in an angle, of which the circular measure is .7854; the value of π being 3.1416.

Ex. 9. The interior angles of a rectilinear figure are in arithmetic progression; the least angle is 120°, and the common difference 5°. Required the number of sides.

Ex. 10. The angles in one regular polygon are twice as many as in another polygon; and an angle of the former: an angle of the latter as 3 : 2. Find the number of sides in each.

Ex. 11. One regular polygon has two sides more than another, and each of its angles exceeds each angle of the other polygon by 15°. Find the number of sides in each.

Ex. 12. The angles of a quadrilateral are in an increasing geometrical progression, and the difference between the third angle and a fourth part of the first angle is 90. Find all the angles.

CHAPTER I.

EXPLANATION OF TERMS. TRIGONOMETRICAL RATIOS.

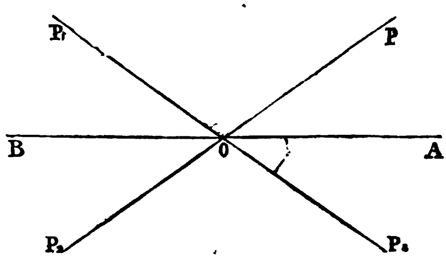
12. If A be any angle, then $\frac{\pi}{2} - A$ or $90^\circ - A$ is called the complement of A, and $\pi - A$ or $180^\circ - A$ is called the supplement of A.

Thus, 30° is the complement of 60°, and 45° the supplement of 135°. Observe, that A must always be *subtracted* from 90° or 180°, whatever be the magnitude of A; thus, the complement of 120° is -30°, since $90^\circ - 120^\circ = -30^\circ$, and the supplement of 215° is -35°, since $180^\circ - 215^\circ = -35^\circ$.

$$\text{Also the complement of } \left(A - \frac{\pi}{2}\right) = \pi - A$$

$$\text{and the supplement of } \left(\frac{\pi}{2} + A\right) = \frac{\pi}{2} - A.$$

13. An angle of 215° has just been mentioned. To understand this we must extend our idea of the magnitude of an angle beyond what has been taught us in ordinary geometry; for there no angle can exceed π, or two right angles.



To do so, conceive a line OP to revolve round the point O, beginning from OA; and let the $\angle AOP = A$;

produce PO to P_2, and through O draw P_1OP_3 perpendicular to PO. Then, when OP has revolved through the $\angle POP_1$, an additional $\frac{\pi}{2}$ has been added to A, or an angle $\frac{\pi}{2} + A$ has been described. Next, let the revolving line come to OP; then another $\frac{\pi}{2}$ has been described, or the whole angle described is $\pi + A$. Let OP continue its revolution till it comes to OP_3, and then another $\frac{\pi}{2}$ must be added, or the whole angle described $= \frac{3\pi}{2} + A$; and when OP describes another right angle, the whole angle will become $2\pi + A$, and the revolving line will again coincide with OP. If OP continue to revolve, then, since in every complete revolution OP describes four right angles, 2π must be added to the former angle for every complete revolution; and thus, if there be n revolutions, the whole angle described will be $2n\pi + A$.

14. We must now remark that the signs + and −, which in arithmetic indicate addition or subtraction, are used in geometry to point out opposition in direction: thus, if a line drawn from a given point as O, towards the right hand, or to A, be called a, an equal line drawn from O towards B would be called $-a$. So also, if the $\angle AOP = \angle AOP_3$, and $AOP = A$, then $AOP_3 = -A$, for $\angle AOP$ may be described by OP revolving from OA upwards, and $\angle AOP_3$ by OP revolving from OA downwards; and thus, also, if lines drawn from P and P_1, perpendicular to AOB, be reckoned positive, lines similarly drawn from P_2 and P_3 are reckoned negative, since their direction is opposed to that of the lines drawn from P and P_1.

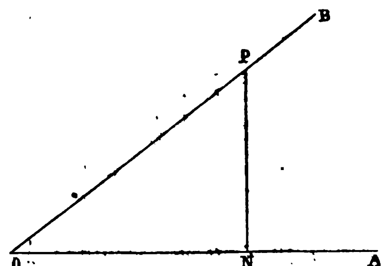
We now proceed to define the trigonometrical ratios.

15. Let AOB be any angle A, in OB take any point P, draw $PN \perp^r OA$; then,

$\frac{PN}{OP}$ is called the sine of A, and is written sin. A.

$\frac{ON}{OP}$ is called the cosine of A, and is written cos. A.

$\frac{PN}{ON}$ is called the tangent of A. and is written tan. A.



Hence, considering ONP to be any right-angled triangle, and A the angle at the base,

$$\sin. A = \frac{\text{altitude}}{\text{hypothenuse}}; \quad \cos. A = \frac{\text{base}}{\text{hypothenuse}};$$

$$\tan. A = \frac{\text{altitude}}{\text{base}}.$$

These are the chief trigonometrical ratios: but there are others, which are sometimes made use of, but which may be shown to be dependent upon the sine and cosine; thus,

$\frac{OP}{ON}$ is the secant of A, or sec. A; $\frac{OP}{PN}$ the cosecant of A, or cosec. A; and $\frac{ON}{PN}$ the cotangent of A, or

cot. A; also $1 - \cos. A$ is called the versed sine of A, or v. sin. A.

16. Since $\angle\ OPN = \frac{\pi}{2} - A$; $\therefore \angle\ OPN$ is the complement of A.

$$\text{And sin. } OPN = \frac{ON}{OP} = \cos. A.$$

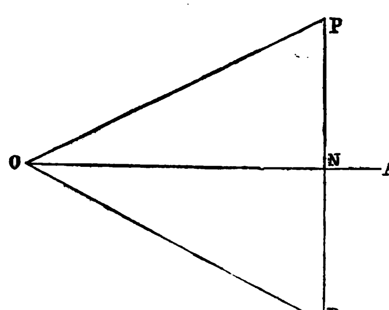
$$\cos. OPN = \frac{PN}{OP} = \sin. A.$$

$$\therefore \sin. \left(\frac{\pi}{2} - A\right) = \cos. A;$$

$$\text{and } \cos. \left(\frac{\pi}{2} - A\right) = \sin. A;$$

or, the sine of an angle is equal to the cosine of its complement, and the cosine of an angle is equal to the sine of its complement.

17. To prove $\sin. (-A) = -\sin. A$;
and $\cos. (-A) = \cos. A$.

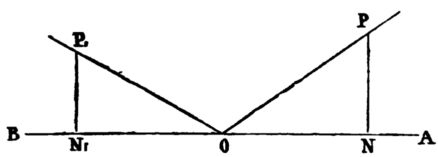


Make $\angle\ AOP_1 = \angle\ AOP$; $\therefore \angle\ AOP_1 = -A$; produce PN to meet OP_1 in P_1; $\therefore OP_1 = OP$, and $P_1N = -PN$.

Now, $\sin. (-A) = \frac{NP_1}{OP_1} = \frac{-NP}{OP} = -\sin. A.$

$$\cos.(-A) = \frac{ON}{OP_1} = \frac{ON}{OP} = \cos. A.$$

18. To prove that $\sin.(\pi - A) = \sin. A$;
and $\cos.(\pi - A) = -\cos. A$.



Make $\angle P_1OB = A$; $\therefore \angle AOP_1 = \pi - A$; make $OP_1 = OP$, draw $P_1N_1 \perp^r OB$.

$$\therefore \sin.(\pi - A) = \frac{P_1N_1}{OP_1} = \frac{PN}{OP} = \sin. A.$$

$$\cos.(\pi - A) = \frac{ON_1}{OP_1} = \frac{-ON}{OP} = -\cos. A;$$

or, the sine of an angle is equal to the sine of its supplement, and the cosine of an angle is equal to the cosine of its supplement, but with a negative sign.

Hence, $\because \frac{\pi}{2} - A$ is the supplement of $\frac{\pi}{2} + A$;

$$\therefore \sin.\left(\frac{\pi}{2} + A\right) = \sin.\left(\frac{\pi}{2} - A\right) = \cos. A \ldots (16.)$$

$$\cos.\left(\frac{\pi}{2} + A\right) = -\cos.\left(\frac{\pi}{2} - A\right) = -\sin. A \ldots (16.)$$

Also, since $-A$ is the supplement of $\pi + A$;

$$\therefore \sin.(\pi + A) = \sin.(-A) = -\sin. A \ \ldots (17.)$$

$$\text{and, } \cos.(\pi + A) = -\cos.(-A) = -\cos. A \ldots (17.)$$

19. To prove that $\sin.(2\pi - A) = -\sin. A$,
and $\cos.(2\pi - A) = \cos. A$.

If (see fig. to art. 17) we conceive the angle AOP constantly to increase till OP coincide with OP_1, then the angle described by the revolving line since it left OA will be $2\pi - A$;

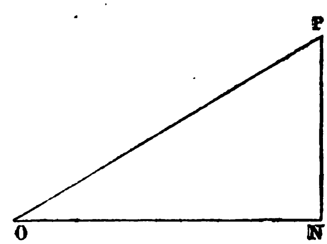
$$\therefore \text{sin.}\,(2\pi - A) = \frac{P_1N}{OP_1} = \frac{-PN}{OP} = -\text{sin.}\,A.$$

$$\text{cos.}\,(2\pi - A) = \frac{ON}{OP_1} = \frac{ON}{OP} = \text{cos.}\,A.$$

20. Finally, since, after each complete revolution of OP, it returns to its original position, and $\therefore$ ON and NP are in the same direction, and of the same magnitude;

$$\therefore \text{sin.}\,(2n\pi + A) = \text{sin.}\,A;$$
$$\text{and cos.}\,(2n\pi + A) = \text{cos.}\,A.$$

21. The sine of an $\angle A$ being given, the cosine, tangent, secant, &c. may be found.



(1.) Let $OP = a$; then, since

$$\frac{PN}{OP} = \text{sin.}\,A, \text{ and } \frac{ON}{OP} = \text{cos.}\,A;$$

$$\therefore PN = a\,\text{sin.}\,A;\ ON = a\,\text{cos.}\,A;$$

$$\therefore PN^2 + ON^2 = a^2 = a^2\,(\text{sin.}^2A + \text{cos.}^2A);$$

$$\therefore \text{sin.}^2A + \text{cos.}^2A = 1;$$

$$\therefore \text{cos.}\,A = \sqrt{1 - \text{sin.}^2A}, \text{ and sin.}\,A = \sqrt{1 - \text{cos.}^2A}.$$

(2.) $\text{Tan.}\,A = \frac{PN}{ON} = \frac{a\sin.A}{a\cos.A} = \frac{\sin.A}{\cos.A}$.

(3.) $\text{Sec.}\,A = \frac{OP}{ON} = \frac{a}{a\cos.A} = \frac{1}{\cos.A}$.

(4.) $\text{Cot.}\,A = \frac{ON}{PN} = \frac{a\cos.A}{a\sin.A} = \frac{\cos.A}{\sin.A}$.

$$\therefore \text{ from (2) } \cot.A = \frac{1}{\tan.A}.$$

(5.) $\text{Cosec.}\,A = \frac{OP}{PN} = \frac{a}{a\sin.A} = \frac{1}{\sin.A}$.

(6.) Also,

$$\text{Sec.}^2A = \frac{OP^2}{ON^2} = \frac{ON^2 + PN^2}{ON^2} = 1 + \frac{PN^2}{ON^2};$$

and, $\frac{PN}{ON} = \tan.A$; $\therefore \sec.A = \sqrt{1 + \tan.^2A}$;

and, $(\text{cosec.}\,A)^2 = \frac{OP^2}{PN^2} = 1 + \frac{ON^2}{PN^2} = 1 + \cot.^2A$;

$$\therefore \text{cosec.}\,A = \sqrt{1 + \cot.^2A}.$$

(7.) And, $\text{v. sin.}\,A = 1 - \cos.A$.

Hence, collecting the results,

$$\text{Sin.}\,A = \sqrt{1 - \cos.^2A}.$$

$$\text{Cos.}\,A = \sqrt{1 - \sin.^2A}.$$

$$\text{Tan.}\,A = \frac{\sin.A}{\cos.A}.$$

$$\text{Sec.}\,A = \frac{1}{\cos.A} = \sqrt{1 + \tan.^2A}.$$

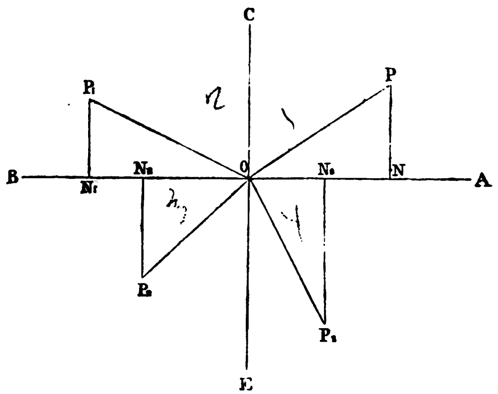
$$\text{Cot.}\,A = \frac{\cos.A}{\sin.A} = \frac{1}{\tan.A}.$$

$$\text{Cosec.}\,A = \frac{1}{\sin.A} = \sqrt{1 + \cot.^2A}.$$

$$\text{V. sin.}\,A = 1 - \cos.A.$$

Which expressions must be carefully remembered.

22. To trace the values of the sine and cosine, while the angle increases from 0 to 2π.



Through a point O draw AOB, COE, at right angles to each other; these lines will divide the angles round the point O into 4 right angles, AOC, COB, BOE, EOA, which we may call the first, second, third, and fourth quadrants, respectively.

In the first quadrant, $\sin. A = \frac{PN}{OP}$; and $\cos. A = \frac{ON}{OP}$; also, PN and ON are both $+$; $\therefore$ the sine and cosine are both $+$. When OP coincides with OA, $\angle A = 0$; $PN = 0$; $ON = OP$; $\therefore \sin. 0 = 0$, and $\cos. 0 = 1$.

As $\angle AOP$ increases, N approaches O; $\therefore ON$ decreases and PN increases; and when OP coincides with OC, $PN = OP$, and $ON = 0$; $\therefore \sin. \frac{\pi}{2} = 1$; $\cos. \frac{\pi}{2} = 0$.

In the second quadrant, $\sin. A = \frac{P_1N_1}{OP_1}$, which is $+$;

but, $\cos. A = \frac{ON_1}{OP_1}$ is $-$; $\because ON_1$ is opposite to ON.

As OP_1 approaches OB, P_1N_1 decreases, and ON increases, and when OP_1 coincides with OB, $P_1N_1 = 0$, and $ON_1 = OP_1$; $\therefore$ from $\frac{\pi}{2}$ to π, the sine decreases, but is $+$, and the cosine increases, but is $-$; also, $\sin. \pi = 0$; $\cos. \pi = -1$.

In the third quadrant, $\sin. A = \frac{P_2N_2}{OP_2}$ and $\cos. A = \frac{ON_2}{OP_2}$, are both negative; the former increases, and the latter decreases; and when OP_2 coincides with OE, or $\angle A = \frac{3\pi}{2}$; $P_2N_2 = OP_2$; $ON_2 = 0$;

$$\therefore \sin. \frac{3\pi}{2} = -1; \cos. \frac{3\pi}{2} = 0.$$

In the fourth quadrant, $\sin. A = \frac{P_3N_3}{OP_3}$ is still negative, but $\cos. A = \frac{ON_3}{OP_3}$ is positive; the former decreases as OP_3 approaches OA, and $= 0$, when OP_3 coincides with OA; the latter increases, and $= 1$ when OP_3 is coincident with OA.

$$\therefore \sin. 2\pi = 0; \text{ and, } \cos. 2\pi = 1.$$

$\therefore$ from 0 to $\frac{\pi}{2}$, sine increases, cosine decreases;

from $\frac{\pi}{2}$ to π, sine decreases, cosine increases;

from π to $\frac{3\pi}{2}$, sine increases, cosine decreases;

from $\frac{3\pi}{2}$ to 2π, sine decreases, cosine increases.

Also, in 1st quadrant, sine is +, cosine is + ;
2d quadrant, sine is +, cosine is − ;
3d quadrant, sine is −, cosine is − ;
4th quadrant, sine is −, cosine is + .

23. To find the values of the tangent and secant in the different quadrants, and their algebraical signs.

Since $\tan. A = \frac{\sin. A}{\cos. A}$, its algebraical sign and value will depend upon those of the sine and cosine; the tangent will be positive, when the sine and cosine have the same sign, and negative when they have contrary signs.

In the 1st quadrant, $\left.\begin{array}{l}\sin.\text{ is } + ;\\ \cos.\text{ is } + ;\end{array}\right\}\ \therefore \tan.\text{ is } + ;$

2d quadrant, $\left.\begin{array}{l}\sin.\text{ is } + ;\\ \cos.\text{ is } - ;\end{array}\right\}\ \therefore \tan.\text{ is } - ;$

3d quadrant, $\left.\begin{array}{l}\sin.\text{ is } - ;\\ \cos.\text{ is } - ;\end{array}\right\}\ \therefore \tan.\text{ is } + ;$

4th quadrant, $\left.\begin{array}{l}\sin.\text{ is } - ;\\ \cos.\text{ is } + ;\end{array}\right\}\ \therefore \tan.\text{ is } - .$

When angle $= 0$, $\left.\begin{array}{l}\sin. = 0;\\ \cos. = 1;\end{array}\right\}\ \therefore \tan. = \frac{0}{1} = 0;$

angle $= \frac{\pi}{2}$, $\left.\begin{array}{l}\sin. = 1;\\ \cos. = 0;\end{array}\right\}\ \therefore \tan. = \frac{1}{0} = \infty;$

angle $= \pi$, $\left.\begin{array}{l}\sin. = 0;\\ \cos. = -1;\end{array}\right\}\ \therefore \tan. = \frac{0}{-1} = 0;$

angle $= \frac{3\pi}{2}$, $\left.\begin{array}{l}\sin. = -1;\\ \cos. = 0;\end{array}\right\}\ \therefore \tan. = -\frac{1}{0} = -\infty;$

angle $= 2\pi$, $\left.\begin{array}{l}\sin. = 0;\\ \cos. = 1;\end{array}\right\}\ \therefore \tan. = \frac{0}{1} = 0.$

Hence, in the first and third quadrants, the tangent is positive, in the second and fourth negative.

It is nothing when the angle is 0, π, or 2π; and is infinite when the angle is $\frac{\pi}{2}$, or $\frac{3\pi}{2}$; but the tangents of these angles have opposite algebraic signs.

24. The secant $= \frac{1}{\text{cos.}}$; it is therefore positive or negative when the cosine is positive or negative.

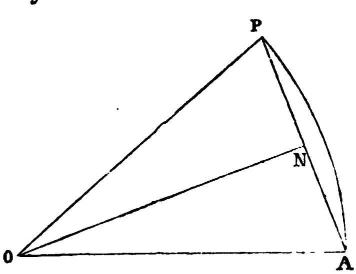
Hence, in the first and fourth quadrants, the secant is +;
in the second and third ...... the secant is −

$$\begin{array}{lll}
\text{When the angle} = 0, & \text{cos.} = 1, & \text{sec.} = 1; \\
\text{angle} = \frac{\pi}{2}, & \text{cos.} = 0, & \text{sec.} = \frac{1}{0} = \infty; \\
\text{angle} = \pi, & \text{cos.} = -1, & \text{sec.} = -1; \\
\text{angle} = \frac{3\pi}{2}, & \text{cos.} = 0, & \text{sec.} = \frac{1}{0} = \infty; \\
\text{angle} = 2\pi, & \text{cos.} = 1, & \text{sec.} = \frac{1}{1} = 1.
\end{array}$$

In the same manner may the algebraical signs and values of the cotangent and cosecant be found from the values of the sines and cosines.

25. The following proposition is frequently useful; the chord of an arc of a circle = twice the radius × sine of $\frac{1}{2}$ the angle subtended by the arc.

Let $\angle\ AOP = A$, AP be an arc round centre O, ANP its chord; draw $ON \perp AP$; $\therefore PN = AN$, and $\angle\ AON = NOP = \frac{A}{2}$;



$$\therefore PNA = 2PN = 2OP \cdot \frac{PN}{OP} = 2OP \cdot \sin. \frac{A}{2}.$$

COR. If we call $OP = r$, and arc $AP = a$, then $A = \frac{a}{r}$;

$$\therefore ANP = 2r \cdot \sin. \frac{1}{2}\left(\frac{a}{r}\right).$$

26. When $\angle AOP = 60^\circ : AP = AO$.

For, $\because OA = OP$; $\therefore \angle OPA = \angle OAP$;

$\therefore 2\angle OPA + 60 = 180$; $\therefore \angle OPA = 60 = \angle AOP$;

$\therefore AP = OA$, or the chord of 60 = radius.

27. We may now find the numerical values of the sine, cosine, tangent, and secant of 30, 45, 60, and 120.

30°.

Since, chord 60 = $2r \cdot \sin.30$; and chord 60 = r;

$$\therefore 2r \sin.30 = r; \quad \therefore \sin.30 = \frac{1}{2};$$

$$\cos.30 = \sqrt{1 - \sin.^2 30} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2};$$

$$\tan.30 = \frac{\sin.30}{\cos.30} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}};$$

$$\sec.30 = \frac{1}{\cos.30} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

45°.

(2.) Since sine of an arc = cosine of its complement;

$$\therefore \sin.45 = \cos.\overline{90 - 45} = \cos.45.$$

But $\sin.^2 45 + \cos.^2 45 = 1$;

$$\therefore 2\sin.^2 45 = 1; \text{ or, } \sin.45 = \frac{1}{\sqrt{2}} = \cos.45;$$

$$\tan. 45 = \frac{\sin. 45}{\cos. 45} = \frac{\sin. 45}{\sin. 45} = 1;$$

$$\sec. 45 = \frac{1}{\cos. 45} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}.$$

60°.

(3.) $$\text{Sin.} 60 = \cos. \overline{90 - 60} = \cos. 30 = \frac{\sqrt{3}}{2};$$

$$\cos. 60 = \sin. \overline{90 - 60} = \sin. 30 = \frac{1}{2};$$

$$\tan. 60 = \frac{\sin. 60}{\cos. 60} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3};$$

$$\sec. 60 = \frac{1}{\cos. 60} = \frac{1}{\frac{1}{2}} = 2.$$

120°.

(4.) Since the sine of an arc = sine of its supplement, and the cosine of an arc = − cosine of its supplement;

$$\sin. 120 = \sin. 60 = \frac{\sqrt{3}}{2};$$

$$\cos. 120 = -\cos. 60 = -\frac{1}{2};$$

$$\tan. 120 = \frac{\sin. 120}{\cos. 120} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3};$$

$$\sec. 120 = \frac{1}{\cos. 120} = +\frac{1}{-\frac{1}{2}} = -2.$$

28. We have already seen (art. 20) that, after each complete revolution of the line OP, the sine and cosine have the same value, or that

$$\sin. A = \sin. (2n\pi + A);$$
$$\cos. A = \cos. (2n\pi + A).$$

Hence the addition of $2n\pi$ to an angle, will make no difference in the sine and cosine; $\therefore$ since

$$\sin. A = \sin. (\pi - A); \text{ and } \cos. A = -\cos. (\pi - A);$$
$$\sin. (2n\pi + A) = \sin. (\overline{2n+1}\,\pi - A);$$
$$\cos. (2n\pi + A) = -\cos. (\overline{2n+1}\,\pi - A).$$

In the same manner, if we suppose the radius OP to revolve in a direction opposite to that which it has hitherto done, and describe angles, $-2\pi + A$, $-4\pi + A$, &c. the sine and cosine will still be the same; or,

$$\sin. A = \sin. (-2n\pi + A); \cos. A = \cos. (-2n\pi + A);$$
$$\sin. A = \sin. (\pm 2n\pi + A); \cos. A = \cos. (\pm 2n\pi + A);$$

or, speaking generally, the addition or subtraction of 2π makes no alteration in the sine and cosine.

Cor. 1. Hence, if $A = 0$; and $\therefore \sin. A = 0$, and $\cos. A = 1$;

$$\sin. 2n\pi = 0; \sin. (2n + 1)\pi = 0;$$
$$\cos. 2n\pi = 1; \cos. (2n + 1)\pi = -1.$$

Cor. 2. If $A = \frac{\pi}{2}$; $\sin. A = 1$; $\cos. A = 0$;

$$\therefore \sin. \left(2n\pi + \frac{\pi}{2}\right) = \sin. (4n + 1). \frac{\pi}{2} = 1;$$
$$\cos. \left(2n\pi + \frac{\pi}{2}\right) = \cos. (4n + 1)\frac{\pi}{2} = 0.$$

29. Again, $\because \sin. A = -\sin. (\pi + A)$;

$$\text{and } \cos. A = -\cos. (\pi + A);$$

$\therefore$ the addition of π to any angle gives the same numerical values to the sine and cosine, but with different algebraic signs; and $\because$ the addition of any multiple of 2π does not make any change;

$$\therefore \sin. (2n\pi + A), \text{ or } \sin. A, = -\sin. (\overline{2n+1}\,\pi + A);$$

$$\cos. (2n\pi + A), \text{ or } \cos. A, = -\cos. (\overline{2n+1}\,\pi + A).$$

30. Again, for negative angles,

$$\because \sin. (-A) = \sin. (2\pi - A) = \therefore \sin. (2n\pi - A);$$

$$\text{and } \cos. (-A) = \cos. (2\pi - A) = \therefore \cos. (2n\pi - A);$$

$$\text{and } \because \sin. A = -\sin. (-A); \ \cos. A = \cos. (-A).$$

$$\therefore \sin. A = -\sin. (2n\pi - A); \ \cos. A = \cos. (2n\pi - A);$$

$\therefore$ collecting the results, $\sin. A = \sin. (\pm 2n\pi + A)$ $= -\sin. (2n\pi - A) = \sin. (\overline{2n+1}\,\pi - A)$ $= -\sin. (\overline{2n-1}\,\pi + A)$.

$$\cos. A = \cos. (\pm 2n\pi + A) = \cos. (2n\pi - A)$$
$$= -\cos. (\overline{2n+1}\,\pi \pm A).$$

EXAMPLES.

$$\text{Sin. } (415) = \sin. (415 - 360) = \sin. 55.$$

$$\text{Sin. } (610) = \sin. (250) = -\sin. 70.$$

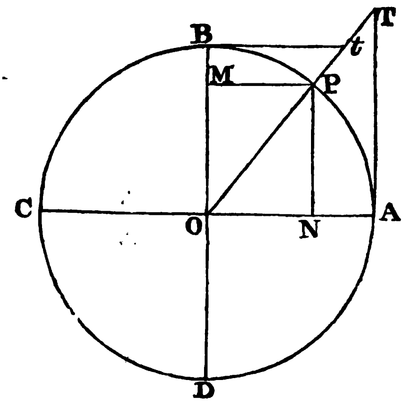
$$\text{Sin. } (-312) = \sin. (360 - 312) = \sin. 48.$$

$$\text{Cos. } (-570) = \cos. (720 - 570) = \cos. 150 = -\cos. 30.$$

31. We may here mention other definitions of the trigonometrical ratios, which were formerly given.

With centre O, and radius OA, represented by unity, describe the circle $ABCD$. Take an arc AP, and let $\angle AOP = A$; draw PN, $AT \perp^r OA$; produce OP to meet AT in T; and similarly draw PM, Bt. Then the lines PN, ON, AT, OT, are called respectively

the sine, cosine, tangent, and secant of A; and Bt, Ot, which are the tangent and secant of the arc BP, or $\angle\ BOP$, which is the complement of AP, or $\angle\ AOP$,



are called the cotangent and cosecant; also AN is called the versed sine of A. These representations of the tangent and secant give a meaning to those terms, which the former definitions do not; and we may see that this mode of representing the trigonometrical ratios by lines can be derived from the methods already given; for

$$\text{sin. } AOP = \frac{PN}{OP} = PN, \text{ if } OP = 1;$$

$$\text{tan. } AOP = \frac{PN}{ON} = \frac{AT}{OA} = AT, \text{ if } OA = 1;$$

$$\text{sec. } AOP = \frac{OP}{ON} = \frac{OT}{OA} = OT, \text{ if } OA = 1;$$

and AT and OT are the tangent and secant of the arc AP.

Also AN, or v. sin. $A = OA - ON = 1 - \text{cos. } A$;

$$\text{and } \angle\ A = \frac{\text{arc } AP}{OA} = \text{arc } AP, \text{ if } OA = 1.$$

Hence the angle and the arc may interchange values.

32. The following examples may be useful:—

(1.) Let sin. A = cos. $4A$; find A.

$$\because \cos. 4A = \sin.(90 - 4A); \therefore \sin. A = \sin.(90 - 4A);$$

$$\therefore A = 90 - 4A; \therefore A = 18^\circ.$$

(2.) Let p sin. $A = q$ cos. A; find sin. A.

$$\therefore p^2 \sin.^2 A = q^2 \cos.^2 A = q^2 . (1 - \sin.^2 A);$$

$$\therefore (p^2 + q^2) \sin.^2 A = q^2; \therefore \sin. A = \frac{q}{\sqrt{p^2 + q^2}}.$$

(3.) Express sin. A in terms of tan. A.

$$\text{Tan. } A = \frac{\sin. A}{\cos. A} = \sin. A \sec. A = \sin. A \sqrt{1 + \tan.^2 A};$$

$$\therefore \sin. A = \frac{\tan. A.}{\sqrt{1 + \tan.^2 A}}.$$

(4.) Let sin. $A = \frac{2}{3}$; find cosine, tangent, and secant.

$$\text{Cos. } A = \sqrt{1 - \sin.^2 A} = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}.$$

$$\text{Tan. } A = \frac{\sin. A}{\cos. A} = \frac{2}{3} \div \frac{\sqrt{5}}{3} = \frac{3}{\sqrt{5}}.$$

$$\text{Sec. } A = \frac{1}{\cos. A} = 1 \div \frac{\sqrt{5}}{3} = \frac{3}{\sqrt{5}}.$$

EXAMPLES.

(1.) Find the sines and cosines of 135°, 150°, 210°, 240°, 315°, 330°, and 480°.

(2.) If sin. $5A$ = sin. A; find A; also, if sin. $2A$ = cos. A.

(3.) If tan. A = 2 sin. A; find A. Also when sin. $3A$ = cos. $2A$.

(4.) If $\tan. A = \frac{4}{3}$; compute the sine and versed sine.

(5.) $6 (\sin. A)^2 = 5 \cos. A$; find sine and cosine.

(6.) $(\text{Sin.} A).(\cos. A) = \frac{\sqrt{3}}{4}$; find sine, cosine, and arc.

(7.) $\text{Tan.} A + \cot. A = \frac{4}{\sqrt{3}}$; find tan. A, and cot. A.

(8.) If $3 \sin. A + 5 \sqrt{3} \cos. A = 9$; find A.

(9.) $25 \sin. A (\sin. A - \cos. A) = 4$; then $\sin. A = \frac{4}{5}$.

(10.) $6 \tan. A + 12 \cot. A = 5 \sqrt{3} \sec. A$;
then $\tan. A = \sqrt{3}$.

(11.) If $\sin. A + \sin. B = \frac{1 + \sqrt{3}}{2}$; and

$\sin. A \ . \ \sin. B = \frac{\sqrt{3}}{4}$; find A and B.

(12.) $\text{Sin.} A = \cos. A \ . \ \tan. A = \frac{1}{\sqrt{1 + (\cot. A)^2}} = \frac{\sqrt{\sec.^2 A - 1}}{\sqrt{\tan.^2 A + 1}}$.

(13.) $\text{Cos.} A = \frac{\sin. A}{\tan. A} = \frac{1}{\sqrt{1 + (\tan. A)^2}} = \frac{\cot. A}{\sqrt{1 + (\cot. A)^2}}$.

(14.) $\text{Tan.} A = \sqrt{\frac{1}{(\cos. A)^2} - 1} = \frac{\sec. A}{\text{cosec.} A} = \frac{\sqrt{1 - (\cos. A)^2}}{\sqrt{1 - (\sin. A)^2}}$.

(15.) Show that $\sec.^2 A \ . \ \text{cosec.}^2 A = \sec.^2 A + \text{cosec.}^2 A$.

(16.) If $m = \tan. A + \sin. A$; $n = \tan. A - \sin. A$; find an equation involving m and n only.

(17.) If $\sin. x \cos. x + a \sin.^2 x = b$; find sin. x
Ex. $a = 7$; $b = 4$; then $x = 45°$.

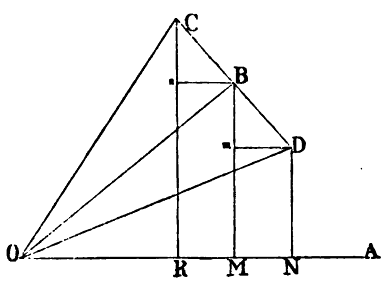
(18.) If $a (\sin. \theta)^2 + b (\cos. \theta)^2 = m$
$b (\sin. \phi)^2 + a (\cos. \phi)^2 = n$
and $a . \tan. \theta = b \tan. \phi$; then $\frac{1}{a} + \frac{1}{b} = \frac{1}{m} + \frac{1}{n}$.

CHAPTER II.

EXPRESSIONS FOR THE SINE, COSINE, AND TANGENT OF $A \pm B$.

33. THE most important proposition in Trigonometry, and upon which all formulas respecting the sum and difference of angles depend, is the following;—

Find sin. $(A \pm B)$, and cos. $(A \pm B)$, in terms of the sines and cosines of A and B.



Let $\angle\ AOB = A$, $\angle\ BOC = \angle\ BOD = B$;

$\therefore\ \angle\ AOC = A + B$; $\angle\ AOD = A - B$.

Through B draw $CBD \perp^{r} OB$; $\therefore\ OC = OD$.

Draw CR, BM, $DN \perp^{r}$ to OA, and Bn, Dm, $\perp^{r}$ to CR, and BM; then the triangles CBn, BmD are similar and equal in every respect; also,

$$\angle\ CBn = \frac{\pi}{2} - nBO = \frac{\pi}{2} - BOA = \frac{\pi}{2} - A.$$

$$\text{Now, sin.}\,(A + B) = \frac{CR}{OC} = \frac{Rn + nC}{OC} = \frac{BM}{OC} + \frac{nC}{OC}.$$

$$\text{But } \frac{BM}{OC} = \frac{BM}{OB} \cdot \frac{OB}{OC} = \text{cos.}\,B\,.\,\text{sin.}\,A;$$

$$\text{and } \frac{Cn}{OC} = \frac{Cn}{CB} \cdot \frac{CB}{CO} = \sin. B \,.\, \cos. A;$$

$$\therefore \sin. (A + B) = \sin. A \,.\, \cos. B + \sin. B \,.\, \cos. A.$$

Again,

$$\sin. (A - B) = \frac{DN}{OD} = \frac{BM - Bm}{OD} = \frac{BM}{OC} - \frac{Cn}{OC}$$

$$= \sin. A \,.\, \cos. B - \sin. B \,.\, \cos. A;$$

$$\text{and } \cos. (A + B) = \frac{OR}{OC} = \frac{OM - RM}{OC} = \frac{OM}{OC} - \frac{Bn}{OC}.$$

$$\text{But } \frac{OM}{OC} = \frac{OM}{OB} \cdot \frac{OB}{OC} = \cos. A \,.\, \cos. B;$$

$$\frac{Bn}{OC} = \frac{Bn}{BC} \cdot \frac{BC}{OC} = \sin. B \,.\, \sin. A;$$

$$\therefore \cos. (A + B) = \cos. A \,.\, \cos. B - \sin. A \,.\, \sin. B.$$

$$\text{Also, } \cos. (A - B) = \frac{ON}{OD} = \frac{OM + Dm}{OD} = \frac{OM}{OC} + \frac{Bn}{OC}$$

$$= \cos A \,.\, \cos. B + \sin. A \,.\, \sin. B.$$

34. Hence, collecting the formulas,

$$\sin. (A + B) = \sin. A \,.\, \cos. B + \sin. B \,.\, \cos. A \,.\,. \quad (1.)$$

$$\sin. (A - B) = \sin. A \,.\, \cos. B - \sin. B \,.\, \cos. A \,.\,. \quad (2.)$$

$$\cos. (A + B) = \cos. A \,.\, \cos. B - \sin. A \,.\, \sin. B \,.\,. \quad (3.)$$

$$\cos. (A - B) = \cos. A \,.\, \cos. B + \sin. A \,.\, \sin. B \,.\,. \quad (4.)$$

35. If $B = A$, we have from (1); $\because A + B = 2A$;

$$\sin. 2A = \sin. A \,.\, \cos. A + \sin. A \cos. A = 2 \sin. A \cos. A.$$

$$\text{and from (3); } \cos. 2A = \cos.^2 A - \sin.^2 A;$$

$$\text{or, } \cos. 2A = \cos.^2 A - (1 - \cos.^2 A) = 2 \cos.^2 A - 1;$$

$$\text{or, } \cos. 2A = 1 - \sin.^2 A - \sin.^2 A = 1 - 2 \sin.^2 A;$$

which formulas give sin. $2A$, and cos. $2A$, in terms of sin. A and cos. A.

36. For $2A$ write A; $\therefore$ for A write $\frac{A}{2}$;

$$\therefore \sin. A = 2 \sin. \frac{A}{2} . \cos. \frac{A}{2};$$

$$\cos. A = 2 \cos.^2 \frac{A}{2} - 1, \text{ or } = 1 - 2 \sin.^2 \frac{A}{2};$$

which express the sine and cosine of an angle, in terms of the sine and cosine of half the angle.

37. Since $2 \sin.^2 \frac{A}{2} = 1 - \cos. A$,

$$\text{and } 2 \cos.^2 \frac{A}{2} = 1 + \cos. A;$$

$$\therefore \frac{\sin.^2 \frac{A}{2}}{\cos.^2 \frac{A}{2}} = \tan.^2 \frac{A}{2} = \frac{1 - \cos. A}{1 + \cos. A};$$

$$\therefore \tan. \frac{A}{2} = \sqrt{\frac{1 - \cos. A}{1 + \cos. A}};$$

a formula of considerable utility.

38 A great variety of formulas may be deduced from the general formulas of Art. 34; the following are the most important;—

Adding 1 and 2 together,

$$\sin. (A + B) + \sin. (A - B) = 2 \sin. A . \cos. B.$$

And similarly also we have,

$$\cos. (A + B) + \cos. (A - B) = 2 \cos. A . \cos. B;$$
$$\sin. (A + B) - \sin. (A - B) = 2 \sin. B . \cos. A;$$
$$\cos. (A - B) - \cos. (A + B) = 2 \sin. A . \sin. B;$$

39. Let $A + B = m$, and $(A - B) = n$;

$$\therefore A = \frac{m+n}{2}, \text{ and } B = \frac{m-n}{2}, \text{ and}$$

$$\sin. m + \sin. n = 2 \sin. \frac{m+n}{2} . \cos. \frac{m-n}{2};$$

$$\cos. m + \cos. n = 2 \cos. \frac{m+n}{2} . \cos. \frac{m-n}{2};$$

$$\sin. m - \sin. n = 2 \sin. \frac{m-n}{2} . \cos. \frac{m+n}{2};$$

$$\cos. n - \cos. m = 2 \sin. \frac{m+n}{2} . \sin. \frac{m-n}{2};$$

40. These last formulas may be obtained directly from the original expressions for sine and cosine of $A \pm B$, by the following method;—

$$\text{Since } A = \frac{A+B}{2} + \frac{A-B}{2} = p + q,$$

$$\text{and } B = \frac{A+B}{2} - \frac{A-B}{2} = p - q;$$

$$\text{where } p = \frac{A+B}{2}, \text{ and } q = \frac{A-B}{2};$$

$$\therefore \sin. A = \sin. \overline{p+q} = \sin. p . \cos. q + \sin. q . \cos. p;$$
$$\cos. A = \cos. \overline{p+q} = \cos. p . \cos. q - \sin. p . \sin. q;$$
$$\sin. B = \sin. \overline{p-q} = \sin. p . \cos. q - \sin. q . \cos. p;$$
$$\cos. B = \cos. \overline{p-q} = \cos. p . \cos. q + \sin. p . \sin. q;$$

therefore,

$$\sin. A + \sin. B = 2 \sin. p . \cos. q = 2 \sin. \frac{A+B}{2} . \cos. \frac{A-B}{2};$$

$$\cos. A + \cos. B = 2 \cos. p . \cos. q = 2 \cos. \frac{A+B}{2} . \cos. \frac{A-B}{2};$$

$$\sin. A - \sin. B = 2 \sin. q \,.\, \cos. p = 2 \sin. \frac{A-B}{2} \,.\, \cos. \frac{A+B}{2};$$

$$\cos. B - \cos. A = 2 \sin. p \,.\, \sin. q = 2 \sin. \frac{A+B}{2} \,.\, \sin. \frac{A-B}{2};$$

41. Hence, also, by division we obtain,

$$\frac{A + \sin. B}{A + \cos. B} = \frac{2 \sin. \frac{A+B}{2} \,.\, \cos. \frac{A-B}{2}}{2 \cos. \frac{A+B}{2} \,.\, \cos. \frac{A-B}{2}} = \frac{\sin. \frac{A+B}{2}}{\cos. \frac{A+B}{2}} = \tan. \frac{A+B}{2};$$

$$\frac{A - \sin. B}{A + \cos. B} = \frac{2 \sin. \frac{A-B}{2} \,.\, \cos. \frac{A+B}{2}}{2 \cos. \frac{A-B}{2} \,.\, \cos. \frac{A+B}{2}} = \frac{\sin. \frac{A-B}{2}}{\cos. \frac{A+B}{2}} = \tan. \frac{A-B}{2};$$

nd,

$$\frac{A + \sin. B}{A - \sin. B} = \frac{2 \sin. \frac{A+B}{2} \,.\, \cos. \frac{A-B}{2}}{2 \cos. \frac{A+B}{2} \,.\, \cos. \frac{A-B}{2}} = \tan. \frac{A+B}{2} \,.\, \cot. \frac{A-B}{2}$$

$$\frac{\tan. \frac{A+B}{2}}{\tan. \frac{A-B}{2}}.$$

42. Again,

$$\text{since } \sin. (A + B) = \sin. A \,.\, \cos. B + \sin. B \,.\, \cos. A;$$
$$\text{and, } \sin. (A - B) = \sin. A \,.\, \cos. B - \sin. B \,.\, \cos. A;$$

therefore,

$$\begin{aligned} \sin. (A+B) \,.\, \sin. (A-B) &= \sin.^2 A \,.\, \cos.^2 B - \sin.^2 B \,.\, \cos.^2 A \\ &= \sin.^2 A \,.\, (1 - \sin.^2 B) - \sin.^2 B \,.\, (1 - \sin.^2 A) \\ &= \sin.^2 A - \sin.^2 B \\ &= (\sin. A + \sin. B) \,.\, (\sin. A - \sin. B); \end{aligned}$$

$$\because \cos. (A+B) = \cos. A \,.\, \cos. B - \sin. A \,.\, \sin. B;$$
$$\text{and } \cos. (A-B) = \cos. A \,.\, \cos. B + \sin. A \,.\, \sin. B;$$

$$\cos.\overline{A+B}\,.\cos.\overline{A-B} = \cos.^2A\,.\cos.^2B - \sin.^2A\,.\sin.^2B$$
$$= \cos.^2A\,.(1-\sin.^2B) - (1-\cos.^2A)\,.\sin.^2B$$
$$= \cos.^2A - \sin.^2B = (\cos.A+\sin.B)\,(\cos.A-\sin.B).$$

43. From Art. 34.

$$\text{Sin.}\,(A+B) = 2\sin.A\,.\cos.B - \sin.\overline{A-B}.$$
$$\text{Cos.}\,(A+B) = 2\cos.A\,.\cos.B - \cos.\overline{A-B}.$$

Hence, if we have three angles, $\overline{A+B}$, A, and $\overline{A-B}$, given in arithmetic progression, the sine and cosine of $\overline{A+B}$ may be found from the sines and cosines of A and $\overline{A-B}$. Thus, to find sin. $\overline{A+B}$, multiply sin. A by 2 cos. B, and sin. $\overline{A-B}$ by (-1), the sum of the products will be sin. $\overline{A+B}$; and to find cos. $\overline{A+B}$, multiply cos. A by 2 cos. B, and cos. $\overline{A-B}$ by (-1), and add the results. The expression 2 cos. $B-1$, is called *the scale of relation*; and sin. $\overline{A+B}$, sin. A, and sin $\overline{A-B}$, are called *terms of a recurring series.*

44. To find sin. $3a$, and cos. $3a$.

$$\begin{aligned}\text{Sin.}\,3a &= \sin.\,(2a+a)\\ &= \sin.\,2a\,.\cos.\,a + \sin.\,a\,.\cos.\,2a\\ &= 2\sin.\,a\cos.^2a + \sin.\,a\,(1-2\sin.^2a)\\ &= 2\sin.\,a - 2\sin.^3a + \sin.\,a - 2\sin.^3a\\ &= 3\sin.\,a - 4\sin.^3a.\end{aligned}$$

$$\begin{aligned}\text{Cos.}\,3a &= \cos.\,(2a+a)\\ &= \cos.\,2a\cos.\,a - \sin.\,2a\sin.\,a\\ &= 2\cos.^3a - \cos.\,a - 2\sin.^2a\cos.\,a\\ &= 2\cos.^3a - \cos.\,a - 2\cos.\,a + 2\cos.^3a\\ &= 4\cos.^3a - 3\cos.\,a.\end{aligned}$$

These formulas show, that, if the sine or cosine of an angle be given, the sine and cosine of a third of the angle can only be found by means of a cubic equation.

45. To find sin. $4a$, and cos. $4a$

$$\begin{aligned}\text{Sin. } 4a &= \sin. 2(2a) = 2\sin. 2a \,.\, \cos. 2a.\\ &= 4\sin. a \,.\, \cos. a \,.\, (1 - 2\sin.^2 a)\\ &= 4\cos. a \,.\, (\sin. a - 2\sin.^3 a).\end{aligned}$$

$$\begin{aligned}\text{Cos. } 4a &= 2\cos.^2 2a - 1\\ &= 2(2\cos.^2 a - 1)^2 - 1\\ &= 8\cos.^4 a - 8\cos.^2 a + 1.\end{aligned}$$

46. Since $\sin. (A+B) + \sin. (A-B) = 2\sin. A \,.\, \cos. B$,
and $\cos. (A+B) + \cos. (A-B) = 2\cos. A \,.\, \cos. B$;

let $B = a$, and $A = (n-1)a$;

$$\therefore A + B = na; \quad \therefore A - B = (n-2)a;$$

$$\therefore \sin. na + \sin. (n-2)a = 2\sin. (n-1)a \,.\, \cos. a,$$
$$\cos. na + \cos. (n-2)a = 2\cos. (n-1)a \,.\, \cos. a.$$

From which formulas the preceding expressions for sin. $3a$, and cos. $3a$ might be found.

Thus, let $n = 3 \quad \therefore \quad n - 2 = 1$;

$$\therefore \sin. 3a + \sin. a = 2\sin. 2a \,.\, \cos. a,$$
$$\therefore \cos. 3a + \cos. a = 2\cos. 2a \,.\, \cos. a.$$

Whence, by putting $2\sin. a \,.\, \cos. a$, for sin. $2a$, and $2\cos.^2 a - 1$, for cos. $2a$; sin. $3a$, and cos. $3a$ may be found in terms of sin. a, and cos. a.

Hence, we may show that

$$\sin. 5a = 16\sin.^5 a - 20\sin.^3 a + 5\sin. a;$$
$$\cos. 5a = 16\cos.^5 a - 20\cos.^3 a + 5\cos. a.$$

47. Find the sine and cosine of an angle in terms of the sine of twice the angle.

$$\cos.^2 A + \sin.^2 A = 1.$$
$$2 \sin. A \,.\, \cos. A = \sin. 2A;$$

$$\therefore\ \cos.^2 A + 2\sin. A \,.\, \cos. A + \sin.^2 A = 1 + \sin. 2A;$$
$$\text{and, } \cos.^2 A - 2\sin. A \,.\, \cos. A + \sin.^2 A = 1 - \sin. 2A;$$

$$\therefore\ \cos. A + \sin. A = \pm \sqrt{1 + \sin. 2A};$$
$$\cos. A - \sin. A = \pm \sqrt{1 - \sin. 2A};$$

$$\therefore\ \cos. A = \tfrac{1}{2}\left\{\sqrt{1 + \sin. 2A} \pm \sqrt{1 - \sin. 2A}\right\}$$
$$\sin. A = \tfrac{1}{2}\left\{\sqrt{1 + \sin. 2A} \mp \sqrt{1 - \sin. 2A}\right\}$$

When A is $\angle$ 45°, cos. A is $>$ sin. A, and we must use the upper signs; but when $A > 45°$, sin. A is $>$ cos. A, and we must use the lower signs.

These values for cos. A and sin. A are termed formulas of verification, since they are used to test the accuracy of results obtained by other processes.

Thus, to find sin. 15°; $\because\ 15° = 45° - 30°$;

$$\therefore\ \sin. 15 = \sin. 45 \,.\, \cos. 30 - \sin. 30 \,.\, \cos. 45$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Then, in the formula, let $A = 15$; $\therefore\ 2A = 30$;

$$\sin. 2A = \frac{1}{2};\ \overline{1 + \sin. 2A} = \frac{3}{2};\ \overline{1 - \sin. 2A} = \frac{1}{2};$$

$$\therefore\ \sin. 15 = \frac{1}{2}\left(\sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}, \text{ as before.}$$

48. To find the tangent of $A \pm B$ in terms of the tangents of A and B.

First,

$$\tan.(A + B) = \frac{\sin. \overline{A+B}}{\cos. \overline{A+B}} = \frac{\sin. A \,.\, \cos. B + \sin. B \,.\, \cos. A}{\cos. A \,.\, \cos. B - \sin. A \,.\, \sin. B}.$$

Divide the numerator and denominator by $\cos. A \,.\, \cos. B$.

The numerator becomes

$$\frac{\sin. A}{\cos. A} + \frac{\sin. B}{\cos. B} = \tan. A + \tan. B.$$

The denominator becomes

$$1 - \frac{\sin. A \,.\, \sin. B}{\cos. A \,.\, \cos. B} = 1 - \tan. A \,.\, \tan. B;$$

$$\therefore \tan. (A + B =) \frac{\tan. A + \tan. B}{1 - \tan. A \,.\, \tan. B}.$$

Secondly,

$$\tan. (A - B) = \frac{\sin. (A - B)}{\cos. (A - B)} = \frac{\sin. A \,.\, \cos. B - \sin. B \,.\, \cos. A}{\cos. A \,.\, \cos. B + \sin. A \,.\, \sin. B};$$

$\therefore$ dividing numerator and denominator by $\cos. A \,.\, \cos. B$,

$$\tan. (A - B) = \frac{\tan. A - \tan. B}{1 + \tan. A \,.\, \tan. B}.$$

COR. 1. In the formula for $\tan. (A + B)$, put $B = A$;

$$\therefore \tan. (A + B) = \tan. 2A = \frac{2 \,.\, \tan. A}{1 - \tan.^2 A}.$$

COR. 2. For $2A$, put A; $\therefore$ for A put $\frac{A}{2}$;

$$\therefore \tan. A = \frac{2 \tan. \frac{A}{2}}{1 - \tan.^2 \frac{A}{2}}.$$

49. Again; if $A = 45$, $\tan. A = \tan. 45 = 1$;

$$\therefore \tan. (45 + B) = \frac{1 + \tan. B}{1 - \tan. B};$$

$$\text{and } \tan. (45 - B) = \frac{1 - \tan. B}{1 + \tan. B};$$

Hence,

$$\tan.(45+B)-\tan.(45-B)=\frac{1+\tan.B}{1-\tan.B}-\frac{1-\tan.B}{1+\tan.B}$$

$$=\frac{(1+\tan.B)^2-(1-\tan.B)^2}{1-\tan.^2B}$$

$$=\frac{4\tan.B}{1-\tan.^2B}=2\tan.2B.$$

And $\tan.4a=\tan.(2a+2a)$

$$=\frac{2\tan.2a}{1-\tan.^2 2a};$$

$\therefore$ $\left(\text{putting for } \tan.2a,\ \frac{2\tan.a}{1-\tan.^2a}\right)$

$$=\frac{4\tan.a\,.\,(1-\tan.^2a)}{(1-\tan.^2a)^2-4\tan.^2a}$$

$$=\frac{4\tan.a-4\tan.^3a}{1-6\tan.^2a+\tan.^4a}.$$

50. Since $\tan.(A+B)=\dfrac{\tan.A+\tan.B}{1-\tan.A\,.\,\tan.B}$,

for B put $\overline{B+C}$;

$$\therefore\ \tan.\overline{A+B+C}=\frac{\tan.A+\tan.\overline{B+C}}{1-\tan.A\,.\,\tan.\overline{B+C}}$$

$$=\frac{\tan.A+\dfrac{\tan.B+\tan.C}{1-\tan.B\,.\,\tan.C}}{1-\tan.A\left(\dfrac{\tan.B+\tan.C}{1-\tan.B\,.\,\tan.C}\right)}$$

$$=\frac{\tan.A+\tan.B+\tan.C-\tan.A\,.\,\tan.B\,.\,\tan.C}{1-(\tan.A\,.\,\tan.B+\tan.A\,.\,\tan.C+\tan.B\,.\,\tan.C)}.$$

Cor. 1. If $A+B+C=\pi$, $\tan.\overline{A+B+C}=0$; therefore, the numerator of the fraction is $=0$; and therefore, $\tan.A+\tan.B+\tan.C=\tan.A\,.\,\tan.B\,.\,\tan.C$.

Hence, if A, B, C, be the $\angle^s$ of a $\triangle$, the number expressed by the sum of the three tangents is equal to the number expressed by their product.

COR. 2. If $A = B = C, A + B + C = 3A$,

$$\text{and } \tan. 3A = \frac{3 \tan. A - \tan.^3 A}{1 - 3 \tan.^2 A}.$$

51. Also, $\sin.(A + B + C) = \sin.(\overline{A + B} + C)$

$= \sin.(A + B) . \cos. C + \sin. C . \cos. \overline{A + B}$

$= \sin. A . \cos. B . \cos. C + \sin. B . \cos. A . \cos. C$

$+ \sin. C . \cos. A . \cos. B - \sin. A . \sin. B . \sin. C.$

52. These are examples of the preceding formulas:

(1.) $\text{Sin.}(45 + A) = \sin. 45 . \cos. A + \sin. A . \cos. 45$

$$= \frac{1}{\sqrt{2}} . (\cos. A + \sin. A).$$

$\text{Cos.}(45 + A) = \cos. 45 . \cos. A - \sin. 45 . \sin. A$

$$= \frac{1}{\sqrt{2}} . (\cos. A - \sin. A).$$

Whence, $\sin. 75 = \cos. 15 = \dfrac{1 + \sqrt{3}}{2\sqrt{2}}$;

$$\cos. 75 = \sin. 15 = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

(2.) $\text{Sin.}(45 + A) - \sin.(45 - A) = \sqrt{2} . \sin. A$;

$\sin.(45 + A) + \sin.(45 - A) = \sqrt{2} . \cos. A.$

$$\therefore \tan. A = \frac{\sin.(45 + A) - \sin.(45 - A)}{\sin.(45 + A) + \sin.(45 - A)}.$$

(3.)* If $A = \tan.^{-1}\frac{1}{2}$; $B = \tan.^{-1}\frac{1}{3}$; then $A + B$ $= 45$;

For $\tan.(A+B) = \dfrac{\tan. A + \tan. B}{1 - \tan. A \,.\, \tan. B} = \dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}$

$$= \frac{\frac{5}{6}}{\frac{5}{6}} = 1 = \tan. 45;$$

$\therefore A + B = 45.$

(4.) Prove that $\tan.(45 - A) = \sqrt{\dfrac{1 - \sin. 2A}{1 + \sin. 2A}}$.

$$\sqrt{1 - \sin. 2A} = \sqrt{\cos.^2 A - 2\sin. A \,.\, \cos. A + \sin.^2 A}$$
$$= \cos. A - \sin. A;$$

$$\sqrt{1 + \sin. 2A} = \sqrt{\cos.^2 A + 2\sin. A \; \cos. A + \sin.^2 A}$$
$$= \cos. A + \sin. A;$$

$$\therefore \sqrt{\frac{1 - \sin. 2A}{1 + \sin. 2A}} = \frac{\cos. A - \sin. A}{\cos. A + \sin. A} = \frac{1 - \tan. A}{1 + \tan. A}$$
$$= \tan.(45 - A).$$

(5.) If $A + B + C = \pi$, show that

$$\sin. A + \sin. B + \sin. C = 4 \,.\, \cos. \frac{A}{2} \,.\, \cos. \frac{B}{2} \,.\, \cos. \frac{C}{2}.$$

$$\because C = \pi - (A + B); \therefore \sin. C = \sin.(A + B);$$

$$\sin. A + \sin. B = 2\sin. \frac{A+B}{2} \,.\, \cos. \frac{A-B}{2};$$

* By $\tan.^{-1}\frac{1}{2}$ is meant an angle of which the tangent is $\frac{1}{2}$, similarly $\sin.^{-1} x$ is an angle whose sine is x.

$$\sin. C = 2 \sin. \frac{A+B}{2} . \cos. \frac{A+B}{2};$$

Also $\sin. \left(\frac{A+B}{2}\right) = \sin. \left(90 - \frac{C}{2}\right) = \cos. \frac{C}{2};$

$\therefore \sin. A + \sin. B + \sin. C$

$$= 2 \cos. \frac{C}{2} . \left\{ \cos. \frac{A+B}{2} + \cos. \frac{A-B}{2} \right\}$$

$$= 4 \cos. \frac{C}{2} . \cos. \frac{A}{2} . \cos. \frac{B}{2} .$$

(6.) To find the sine and cosine of 18° and 36°.

$5 \times 18^\circ = 90^\circ; \ \therefore 3 \times 18^\circ = 90^\circ - 2 \times 18^\circ;$

$\therefore \cos. (3 \times 18^\circ) = \cos. (90^\circ - 2 \times 18^\circ) = \sin. \overline{2 \times 18^\circ}.$

But $\cos. \overline{3 \times 18} = 4 \cos.^3 18 - 3 . \cos. 18,$

$\sin. \overline{2 \times 18} = 2 \sin. 18 . \cos. 18;$

$\therefore 4 \cos.^3 18 - 3 \cos. 18 = 2 \sin. 18 . \cos. 18,$

or $4 \cos.^2 18 - 3 = 2 \sin. 18.$

Let $\sin. 18 = x;$

$\therefore 4(1 - x^2) - 3 = 2x$, or $4x^2 + 2x = 1;$

$$\therefore x^2 + \frac{x}{2} + \frac{1}{16} = \frac{1}{4} + \frac{1}{16} = \frac{5}{16};$$

$$\therefore x = \frac{\sqrt{5} - 1}{4} = \sin. 18 = \cos. 72.$$

$$4(1 - x^2) = 4 \cos.^2 18 = 2x + 3 = \frac{\sqrt{5} - 1}{2} + 3 = \frac{5 + \sqrt{5}}{2}.$$

$$\therefore \sqrt{1 - x^2} = \cos. 18 = \frac{\sqrt{5 + \sqrt{5}}}{2\sqrt{2}};$$

$$\sin. 36 = 2 \sin. 18 \times \cos. 18 = \frac{(\sqrt{5} - 1) \ \sqrt{5 + \sqrt{5}}}{4\sqrt{2}}$$

$$=\frac{\sqrt{(6-2\sqrt{5})(5+\sqrt{5})}}{4\sqrt{2}}=\frac{\sqrt{20-4\sqrt{5}}}{4\sqrt{2}}=\frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}};$$

$$\cos.36=1-2\sin.^2 18=1-\frac{(\sqrt{5}-1)^2}{8}=1-\frac{6-2\sqrt{5}}{8}$$

$$=\frac{2+2\sqrt{5}}{8}=\frac{1+\sqrt{5}}{4}.$$

EXAMPLES.

(1.) Find the sines and cosines of 105°, and 33°.

(2.) $\text{Cos.}\,(60+A)+\cos.\,(60-A)=\cos.\,A;$

$\cos.\,(60-A)-\cos.\,(60+A)=\sqrt{3}\,.\,\sin.\,A;$

$\sin.\,(30+A)+\sin.\,(30-A)=\cos.\,A;$

$\cos.\,(30-A)-\cos.\,(30+A)=\sin.\,A.$

(3.) If $\tan.\,2A=3\tan.\,A$, find A; and $\sin.\,A$, when $\cos.\,2A=2\cos.\,A$.

(4.) $$\text{Sin.}\,A=\frac{2\tan.\,\frac{A}{2}}{1+\tan.^2\frac{A}{2}}=\frac{2}{\tan.\,\frac{A}{2}+\cot.\,\frac{A}{2}}$$

$$=1-2\sin.^2\left(45-\frac{A}{2}\right).$$

(5.) $$\text{Cos.}\,A=\frac{\cot.\,\frac{A}{2}-\tan.\,\frac{A}{2}}{\cot.\,\frac{A}{2}+\tan.\,\frac{A}{2}}=\frac{1}{1+\tan.\,A\,.\,\tan.\,\frac{A}{2}}$$

$$=2\cos.\left(45+\frac{A}{2}\right).\cos.\left(45-\frac{A}{2}\right).$$

(6.) $\tan. A = \dfrac{2\cot.\frac{A}{2}}{\cot.^2\frac{A}{2} - 1} = \dfrac{2}{\cot.\frac{A}{2} - \tan.\frac{A}{2}}$

$$= \frac{1}{2}\left\{\tan.\left(45 + \frac{A}{2}\right) - \tan.\left(45 - \frac{A}{2}\right).\right\}$$

(7.) If $A = \sin.^{-1}\frac{3}{5}$, and $B = \sin.^{-1}\frac{4}{5}$; $A + B = 90°$.

(8.) If $A = \tan.^{-1}\frac{1}{7}$, and $B = \tan.^{-1}\frac{1}{3}$; $A + 2B = 45°$.

(9.) If $\tan. 3A = 5\tan. A$, find $\tan. A$; and if $\sin. 3A = 2\sin. A$, find A.

(10.) If $\sin. A + \cos. 2A = \sqrt{\frac{5}{4}}$, find A.

(11.) Compute tan. 15°, and show that sin. 30° is a mean proportional between sin. 18° and sin. 54°.

(12.) If $\cos. A = \sqrt{\frac{2}{3}}$; $\cos. B = \dfrac{\sqrt{3} + \sqrt{2}}{2\sqrt{3}}$;

$A + B = 60°$.

(13.) If $\cos. A + \cos. B = \frac{1}{2}(\sqrt{2} + \sqrt{3})$,

and $\cos. 3A + \cos. 3B = -\dfrac{1}{\sqrt{2}}$;

$A = 30$; $B = 45$.

(14.) If $\sin. 2x = (\sin. 3x)^2$; find x.

(15.) If $4\sin. a \,.\, \sin. 3a = 1$; $a = 18°$.

(16.) $\sin. 3a = 4\sin. a \,.\, \sin. (60 - a) \,.\, \sin. (60 + a)$

$\cos. 3a = 4\cos. a \,.\, \sin. (30 - a) \,.\, \sin. (30 + a)$

(17.) Show that $2\cos. 11^\circ 15' = \sqrt{2+\sqrt{2+\sqrt{2}}}$;

$$4\sin. 9^\circ = \sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}};$$

$$\text{and } \tan. 22^\circ 30' = \sqrt{2} - 1.$$

(18.) Prove that $\cos. A = \dfrac{1-\tan.^2\frac{A}{2}}{1+\tan.^2\frac{A}{2}}$;

$$\tan. A = \cot. A - 2\cot. 2A;$$

$$\text{cosec. } A = \cot. \frac{A}{2} - \cot. A.$$

(19.) $\text{Tan. } A = \sqrt{\dfrac{1-\cos. 2A}{1+\cos. 2A}}$.

(20.) $\text{Tan.}^2 \dfrac{A}{2} = \dfrac{2\sin. A - \sin. 2A}{2\sin. A + \sin. 2A}$.

(21.) $\text{Cot.}^2\left(45 + \dfrac{A}{2}\right) = \dfrac{2\,\text{cosec. } 2A - \sec. A}{2\,\text{cosec. } 2A + \sec. A}$.

(22.) If $A + B + C = 90$,

$$\cot. A + \cot. B + \cot. C = \cot. A \,.\, \cot. B \,.\, \cot. C;$$

(23.) If $A + B + C = 180$,

$$\sin. 2A + \sin. 2B + \sin. 2C = 4\sin. A \,.\, \sin. B \,.\, \sin. C.$$

(24.) Find $\sin. A$ from $\tan. \dfrac{A}{2} = \text{cosec. } A - \sin. A$;

(25.) If $\tan. A + 2\cot. 2A$

$$= \sin. A\left(1 + \tan. A \,.\, \tan. \frac{A}{2}\right) . \text{ find } A.$$

(26.) If $A, B, C,$ be in arithmetical progression.

$$\text{Sin. } A - \sin. C = 2\sin. (A - B) \,.\, \cos. B.$$

CHAPTER III.

CONSTRUCTION OF TRIGONOMETRICAL TABLES.

53. THE principal use of the formulas which have been proved in the preceding chapters, is for the purpose of forming trigonometrical tables.

In these tables the values of the sines, cosines, tangents, cotangents, &c. of all angles, from 0° to 45°, are registered. And since $\sin.\overline{45 - A} = \cos.\overline{45 + A}$, and $\cos.\overline{45 - A} = \sin.\overline{45 + A}$, the values of all the trigonometrical lines, from 45° to 90°, are also known.

54. If $\sin. a$ be known, $\cos. a = \sqrt{1 - \sin.^2 a}$ is also known; and $\tan. a = \frac{\sin. a}{\cos. a}$, $\sec. a = \frac{1}{\cos. a}$, $\cot. a = \frac{1}{\tan a}$, $\operatorname{cosec}. a = \frac{1}{\sin. a}$, may be found.

Also, since $\sin. (a + b) = \sin. a . \cos. b + \sin. b . \cos. a$,

and $\cos. (a + b) = \cos. a . \cos. b - \sin. a . \sin. b$.

By making $b = a, 2a, 3a$, &c., $\sin. 2a$, $\cos. 2a$, $\sin. 3a$, $\cos. 3a$, $\sin. 4a$, $\cos. 4a$, &c., may be computed.

Thus, if $\sin. 1'$ be known, $\sin. 2'$, $\sin. 3'$, $\cos. 2'$, $\cos. 3'$, &c., may be found.

55. The $\sin. 1'$ differs but little from $1'$, and may be taken equal to it; for the error arising from this assumption is less than the cube of the decimal which represents a minute.

56. Now, since in a right angle, or $\frac{\pi}{2}$, there are 90×60 minutes, and that $\pi = 3.1415926$;

$$\therefore\ 1' = \frac{3.1415926}{180 \times 60} = \frac{.031415926}{108} = .00029088, \text{ \&c.};$$

$$\therefore\ \sin.\ 1' = .000290888, \text{ \&c.}$$

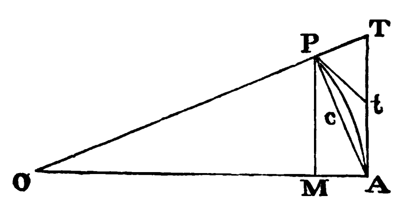
$$\cos.\ 1' = \sqrt{1 - \sin.^2 1'} = .99999995, \text{ \&c.}$$

57. The limit of the error committed in assuming the $\sin.\ 1' = 1'$, may be thus found.

Lemma. If a be $< \frac{\pi}{2}$, $a \angle \tan.\ a,\ 7 \sin.\ a$.

Let AP be a circular arc, AcP its chord, AT a tangent, Pt a tangent at P;

$$\therefore\ Pt = At, \text{ and } \angle\ AOt = \angle\ tOP.$$



Then, manifestly, the arc AP is $\angle\ At + Pt$, $\angle\ At + Tt$, or $\angle\ AT$, (since $TPt = 90°$), but is

$$7 \text{ chord } AcP; \text{ and } \therefore\ 7\ PM;$$

$$\therefore\ \frac{AP}{OA} \text{ is } < \frac{AT}{OA} > \frac{PM}{OA};$$

$$\therefore\ a \angle \tan.\ a\ .\ 7 \sin.\ a.$$

Hence, since tan. a, or $\frac{\sin. a}{\cos. a} > a$;

$\therefore$ sin. $a > a \,.\, \cos. a$, and $\therefore$ *à fortiori*, $> a \,.\, \cos.^2 a$.

For cos. a is a fraction < 1, and therefore diminishes the quantity into which it is multiplied;

$$\therefore \sin. a > a - a \sin.^2 a;$$

$$\therefore a \,.\, \sin.^2 a > a - \sin. a.$$

But sin. $a < a$; therefore $a \times \sin.^2 a < a^3$;

$\therefore$ *à fortiori*, $a^3 > a - \sin. a$, or $a - \sin. a < a^3$.

Hence the error is less than $(.000290888)^3$, or .000000000008 nearly, where the first significant figure is the 12th from the decimal point.

58. The sin. 1′ has been found by assuming that sin. 1′ = 1′; the accuracy of the result may be tried by obtaining the value by a different process.

We have seen that $\sin. 5a = 5 \sin. a - 20 \sin.^3 a + 16 \sin.^5 a$, and sin. 15 is known $= \frac{\sqrt{3} - 1}{2\sqrt{2}}$. Assume, therefore, $5a = 15$, and therefore $a = 3$, and we shall be able to find sin. a, or sin. 3°; but from $\sin. 3a = 3 \sin. a - 4 \sin.^3 a$, we may calculate sin. a = sin. 1°; and again, by trisection, we find sin. 20′, then sin. 10′, and sin. 2′, and lastly sin. 1′.

59. To find sine and cosine 2′, 3′, 4′, &c.

Since $\sin. (n + 1) b + \sin. (n - 1) b = 2 \sin. nb \,.\, \cos. b$;
and $\cos. (n + 1) b + \cos. (n - 1) b = 2 \cos. nb \,.\, \cos. b$.

$\therefore \sin. (n + 1) b = 2 \sin. nb \,.\, \cos. b - \sin. (n - 1) b$;
and $\cos. (n + 1) b = 2 \cos. nb \,.\, \cos. b - \cos. (n - 1) b$.

Let $b = 1'$, and $n = 1, 2, 3$, &c.

$\text{Sin. } 2' = 2 \sin. 1' \,.\, \cos. 1' - \sin. 0 = 2 \sin. 1' \,.\, \cos. 1'$.
$\text{Cos. } 2' = 2 \cos. 1' \,.\, \cos. 1' - \cos. 0 = 2 \cos.^2 1' - 1$.

$$\sin. 3' = 2 \sin. 2' \,.\, \cos. 1' - \sin. 1'.$$
$$\cos. 3' = 2 \cos. 2' \,.\, \cos. 1' - \cos. 1'.$$
$$\sin. 4' = 2 \sin. 3' \,.\, \cos. 1' - \sin. 2'.$$
$$\cos. 4' = 2 \cos. 3' \,.\, \cos. 1' - \cos. 2'.$$
$$\&c. \ldots = \&c. \ldots\ldots$$

60. Also, since the sin. $1°$ is known, we may by the same formulas, by putting $b = 1$, compute the sines and cosines of all the other angles.

61. Next, to find the sines and cosines of angles composed of degrees and minutes.

$$\text{Sin.}\,(a + b) = 2 \sin. a \,.\, \cos. b - \sin. (a - b)$$

$$= 2 \sin. a \,.\, \left(1 - 2 \sin.^2 \frac{b}{2}\right) - \sin. (a - b),$$

putting for cos. b its value, $1 - 2 \sin.^2 \frac{b}{2}$;

$$\therefore \sin. (a + b) = 2 \sin. a - \sin. (a - b) - 4 \sin. a \,.\, \sin.^2 \frac{b}{2}.$$

$$= \sin. a + (\sin. a - \sin. \overline{a - b}) - 4 \sin. a \,.\, \sin.^2 \frac{b}{2}.$$

Let $b = 1'$. Therefore,

$$\sin. (a + 1') = \sin. a + (\sin. a - \sin. \overline{a - 1'}) - 4 \sin. a \,.\, \sin.^2 30''.$$

Thus, if we wish to compute the sines of $5°\ 1'$, $5°\ 2'$, &c. we must write for a, $5°$, $5°\ 1'$, &c.

$$\text{Sin.}\, 5°\ 1' = \sin. 5° + (\sin. 5° - \sin. 4° 59') - 4 \sin. 5_{0} \,.\, \sin.^2 30'';$$
$$\sin. 5°\ 2' = \sin. 5°\ 1' + (\sin. 5°\ 1' - \sin. 5°) - 4 \sin. 5°\ 1' \,.\, \sin.^2 30'';$$

and so on.

62. After computing as far as $60°$, the sines of the remaining angles up to $90°$ may be found from the values of the sines previously obtained. For we have seen that $\sin. (60 + A) = \sin. A + \sin. (60 - A.)$

Thus, let $A = 1°, 2°, 3°$, &c.;

$$\therefore\ \sin. 61° = \sin. 1° + \sin. 59°;$$
$$\sin. 62° = \sin. 2° + \sin. 58°.$$

And thus the sines of the remaining angles are known by addition.

63. Since $\cos. A = \sin. \overline{90 - A}$, the cosines will be known if the sines of the angle, from 0 to 90, are computed; and from the sines and cosines the values of the other trigonometrical lines may be found.

64. After computing the tangents up to 45°, from the formula $\tan. A = \frac{\sin. A}{\cos. A}$, the remaining tangents may be found by addition only.

$$\text{For } \tan. \overline{45 + A} - \tan. \overline{45 - A} = 2 \tan. 2A; \text{ (Art. 49)}$$
$$\therefore\ \tan. \overline{45 + A} = 2 \tan. 2A + \tan. \overline{45 - A}.$$

Let $A = 1°, 2°, 3°$, &c.;

$$\begin{aligned} \therefore\ \tan. 46° &= 2 \tan. 2° + \tan. 44°. \\ \tan. 47° &= 2 \tan. 4° + \tan. 43°. \\ \tan. 48° &= 2 \tan. 6° + \tan. 42°. \\ \&c.\ldots &= \&c.\ldots\ldots \end{aligned}$$

65. By these and similar methods, the values of the sines, tangents, &c. are found; these are called natural sines, tangents, &c. In practice, however, the logarithms of these numbers are more convenient; but since the sines and cosines are always less than unity, all their logarithms will be negative. Also the tangent of an angle less than 45 being less than unity, and of an angle greater than 45° being greater than unity, the logarithms of the tangents up to 45° would be negative, and of angles greater than 45° would be positive. To avoid the confusion which would thus take place in the tables,

the logarithms of the trigonometrical ratios are increased by 10; and we must therefore remember, in practice, to subtract 10 from each logarithmic sine or cosine, &c.

66. Many of the values for the sines and tangents are found by addition only; consequently, if there be any error in one of the quantities, this error will be repeated as many times as the operation is repeated. To guard against the possibility of such a circumstance happening, and for the purpose of detecting the error so arising, formulas have been invented, which will give the required result, by a different and independent process. These formulas are, from the use made of them, called Formulas of Verification. Such a one is $\sin. a = \frac{1}{2}(\sqrt{1+\sin. 2a} - \sqrt{1-\sin. 2a})$. Let us apply this to verify the value obtained for sin. 1°.

$$\text{Let } 2a = 18°; \ \therefore\ a = 9°, \text{ and } \sin. 2a = \frac{\sqrt{5}-1}{4};$$

$$1+\sin. 2a = \frac{3+\sqrt{5}}{4}, \text{ and } 1-\sin. 2a = \frac{5-\sqrt{5}}{4};$$

therefore,

$$\sin. 9° = \tfrac{1}{4}\{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\} = \tfrac{1}{4}\left\{\frac{\sqrt{5}+1}{\sqrt{2}} - \sqrt{5-\sqrt{5}}\right\}.$$

The value of sin. 9° being known, sin. 3° may be found from the formula for sin. $3a$; and, knowing sin. 3°, we may by the same formula find sin. 1°.

67. The following formulas, which depend upon sin. 36° and sin. 72°, have been given by Euler.

$$\text{Since } \cos. 36 = \frac{\sqrt{5}+1}{4}, \text{ and } \cos. 72 = \sin. 18 = \frac{\sqrt{5}-1}{4};$$

also,

$$\sin.\overline{36+A}-\sin.\overline{36-A}=2\cos.36\,.\,\sin.A=\frac{\sqrt{5}+1}{2}\,.\,\sin.A\,;$$

$$\sin.\overline{72+A}-\sin.\overline{72-A}=2\cos.72\,.\,\sin.A=\frac{\sqrt{5}-1}{2}\,.\,\sin.A.$$

Subtracting the lower equation from the upper,

$$\sin.(36+A)-\sin.(36-A)+\sin.\overline{72-A}-\sin.\overline{72+A}=\sin.A\,;$$

$$\therefore\ \sin.A+\sin.\overline{36-A}+\sin.\overline{72+A}=\sin.\overline{36+A}+\sin.\overline{72-A}.$$

In a similar manner it may be shown, that

$$\cos.A+\cos.\overline{72+A}+\cos.\overline{72-A}=\cos.\overline{36+A}+\cos.\overline{36-A}\,.$$

Thus, let $A = 1°$;

$\therefore$ sin. 1° + sin. 35° + sin. 73 = sin. 37° + sin. 71°,
and cos. 1° + cos. 73° + cos. 71° = cos, 37° + cos. 35°,

and if upon trial these equations be not verified, there must have been an error in computing the sines or cosines of the angles.

Example. By reference to Hutton's Tables, we find,

sin. 1° = .0174524	sin. 37 = .6018150
sin. 35° = .5735764	sin. 71° = .9455186
sin. 73° = .9563048	1.5473336
1.5473336	

For further details on this subject, the student is referred to the article on Trigonometry, in the Encyclopædia Metropolitana, by Professor Airy, and to the Appendix to Woodhouse's Trigonometry.

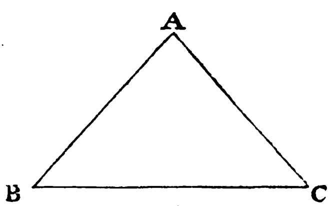
CHAPTER IV.

SOLUTION OF TRIANGLES. EXPRESSIONS FOR THE SINE AND COSINE OF THE ANGLE IN TERMS OF THE SIDES.

68. The parts of a triangle are six, three angles and three sides; and of these any three being given, except the three angles, the remaining parts may be found.

This proposition may be illustrated by geometrical constructions.

First. Let there be given two angles, and one side adjacent to the given angles.



Let BC be taken = the given side.
At the point B, make $\angle\ CBA$ = one given angle.
. . . . C, make $\angle\ BCA$ = the other given angle.
Then the lines CA and BA meet in A; and the vertex of the triangle is determined, and the triangle is constructed.

Secondly. Let two sides, and the angle included by them, be known.

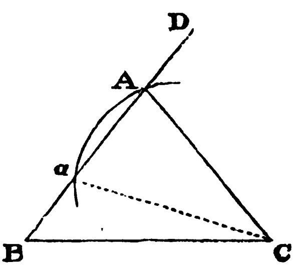
Let BC be one side; make $\angle\ CBA$ = the given angle, and BA = the other given side; join AC; it is the remaining side, and the triangle is constructed.

Thirdly. Let the three sides be given.
Take BC = one of them, and with centre B and radius

BA, equal to another of the sides, describe a circular arc; and with centre *C* and radius *CA*, equal to the remaining side, describe another arc, cutting the former in *A*; *A* is the vertex of the triangle.

Fourthly. If two sides, and an angle opposite to one of them, be given.

Take *BC*, the given side, which is not opposite the given angle, and $\angle$ *CBD*, equal the given angle; with centre *C* and radius *CA*, equal the other side, describe an arc, cutting *BD* in *A*; *ABC* is the Δ required. There is, however, in this case, an ambiguity when *CA* is $\angle$ *CB*, and the angle *B* acute; for then the circular arc, of which *C* is the centre, will meet



AB again in some point (*a*), and it will be doubtful whether Δ *ABC*, or Δ *aBC*, be the one required, as both fulfil all the requisite conditions.

Fifthly. If the three angles be given, innumerable triangles may be drawn, having their angles the same as the given angles. For if *ABC* be a triangle, having its angles *A*, *B*, *C*, equal the given angles, we may, by drawing lines parallel to *AB*, *BC*, and *AC*, form as many triangles as we wish. Hence the magnitude of a triangle cannot be determined, if the three angles be alone given.

69. We proceed to give methods by which the required lines and angles may be computed; but we must previously establish the truth of a few propositions, which exhibit the relations that exist between the sides and angles of a triangle.

Relation between the Angles and Sides of the Triangle.

70. In every triangle, the sines of the angles are proportional to the sides which subtend the angles.

Let ABC be a triangle, AD a perpendicular from A, upon the base, or, it produced; and let A, B, C, represent the three angles, and a, b, c, the three sides opposite the angles.

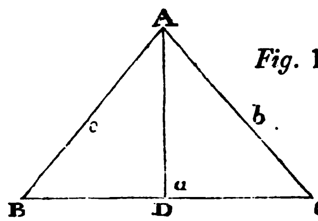


Fig. 1.

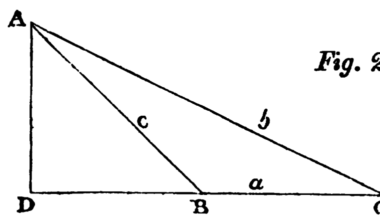


Fig. 2.

Then, in *Fig.* 1,

$$\sin. B = \frac{AD}{c}, \text{ and } \sin. C = \frac{AD}{b};$$

$$\therefore \frac{\sin. B}{\sin. C} = \frac{AD}{c} \times \frac{b}{AD} = \frac{b}{c}.$$

And, in *Fig.* 2,

$$\sin. ABD = \sin. (\pi - B) = \sin. B = \frac{AD}{c}, \ \sin. C = \frac{AD}{b};$$

$$\therefore \frac{\sin. B}{\sin. C} = \frac{AD}{c} \times \frac{b}{AD} = \frac{b}{c}, \text{ as before;}$$

$$\text{or, } \sin. B : \sin. C :: b : c.$$

And by drawing a perpendicular from B, upon AC, we may show that $\sin. A : \sin. C :: a : c$, whence we immediately obtain, $\sin. B : \sin. A :: b : a$.

71. To find the cosine of the angle of a triangle in terms of the sides.

In *Fig.* 1. $AC^2 = AB^2 + BC^2 - 2BC \,.\, BD$. Euclid, Book II. Prop. 13.

In *Fig.* 2. $AC^2 = AB^2 + BC^2 + 2BC \,.\, BD$. Euclid, Book II. Prop. 12.

But in *Fig.* 1. $\frac{BD}{BA} = \cos. B$, or $BD = BA \,.\, \cos. B$.

.... *Fig.* 2. $\frac{BD}{BA} = \cos. ABD = \cos.(\pi - B) = -\cos. B$;

$$\therefore \; BD = -BA \,.\, \cos. B.$$

Therefore, in both cases,

$$AC^2 = AB^2 + BC^2 - 2AB \,.\, BC \,.\, \cos. B;$$

$$\text{or, } b^2 = c^2 + a^2 - 2ca \,.\, \cos. B;$$

$$\therefore \; \cos. B = \frac{a^2 + c^2 - b^2}{2ac}.$$

For B write A, for b, a, and for a, b, then

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc},$$

similarly,

$$\cos. C = \frac{a^2 + b^2 - c^2}{2ab}.$$

72. To find the sine of the angle of a triangle in terms of the sides.

$$\sin.^2 A = (1 - \cos.^2 A) = (1 - \cos. A) \,.\, (1 + \cos. A).$$

But, $1 - \cos. A = 1 - \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{2bc - b^2 - c^2 + a^2}{2bc}$

$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} = \frac{a^2 - \overline{b - c}|^2}{2bc} = \frac{(a + b - c)\,.\,(a - b + c)}{2bc}.$$

And, $1 + \cos. A = 1 + \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{b^2 + 2bc + c^2 - a^2}{2bc}$.

$$= \frac{(b + c)^2 - a^2}{2bc} = \frac{(b + c + a)\,.\,(b + c - a)}{2bc};$$

$$\therefore \sin.^2 A = \frac{(a + b + c)\,.\,(a + b - c)\,.\,(a + c - b)\,.\,(b + c - a)}{4\,b^2\,c^2}.$$

Let $(a + b + c) = 2S$; or let $S =$ semi-perimeter.

$$\therefore\ a + b - c = 2S - 2c = 2(S - c);$$

$$(a + c - b) = 2S - 2b = 2(S - b);$$

$$(c + b - a) = 2S - 2a = 2(S - a);$$

$$\therefore \sin.^2 A = \frac{16S\,.\,(S - a)\,.\,(S - b)\,.\,(S - c)}{4b^2c^2}.$$

$$\therefore \sin. A = \frac{2}{bc}\,.\,\sqrt{S\,.\,(S - a)\,.\,(S - b)\,.\,(S - c)}.$$

In the same manner sin. B and sin. C may be found; but they may be readily derived from sin. A.

For B put A, for b, a, and for a, b;

$$\therefore \sin. B = \frac{2}{ac}\,.\,\sqrt{S\,.\,(S - a)\,.\,(S - b)\,.\,(S - c)},$$

and similarly,

$$\sin. C = \frac{2}{ab}\,.\,\sqrt{S\,.\,(S - a)\,.\,(S - b)\,.\,(S - c)}.$$

73. Given two sides, and the included angle, to find the area of a triangle.

The area of a triangle is half the rectangle on the same base and of the same altitude, and is therefore equal to half the product of the base into the altitude. Hence

$$\Delta\, ABC = \frac{BC \times AD}{2} = \frac{ac \sin. B}{2};$$

or the area of a triangle is equal to half the product of any two sides multiplied by the sine of the angle included by them.

74. To find the area of a triangle in terms of the sides.

$$\text{Since } \sin. B = \frac{2}{ac} \sqrt{S.(S-a).(S-b).(S-c)};$$

$$\therefore \text{ area} = \frac{ac}{2} \sin. B = \sqrt{S.(S-a).(S-b).(S-c)},$$

which gives the area of a triangle in terms of its sides.

75. In every triangle show that $\dfrac{a+b}{a-b} = \dfrac{\tan. \frac{A+B}{2}}{\tan. \frac{A-B}{2}}$.

$$\text{For } \frac{a}{b} = \frac{\sin. A}{\sin. B}.$$

Add and subtract unity from each side successively;

$$\therefore \frac{a+b}{b} = \frac{\sin. A + \sin. B}{\sin. B},$$

$$\text{and } \frac{a-b}{b} = \frac{\sin. A - \sin. B}{\sin. B};$$

$$\therefore \frac{a+b}{a-b} = \frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{2 \sin. \frac{A+B}{2} . \cos. \frac{A-B}{2}}{2 \sin. \frac{A-B}{2} . \cos. \frac{A+B}{2}}$$

$$= \frac{\sin. \frac{A+B}{2}}{\cos. \frac{A+B}{2}} \times \frac{\cos. \frac{A-B}{2}}{\sin. \frac{A-B}{2}}$$

$$= \frac{\tan. \frac{A+B}{2}}{\tan. \frac{A-B}{2}}.$$

COR. Since $A + B = 180 - C$; $\therefore \frac{A+B}{2} = 90 - \frac{C}{2}$,

$$\text{and } \tan. \frac{A+B}{2} = \cot. \frac{C}{2}.$$

$$\therefore \frac{a+b}{a-b} = \frac{\cot. \frac{C}{2}}{\tan. \frac{A-B}{2}} = \cot. \frac{C}{2} \times \cot. \frac{A-B}{2}.$$

From this formula, knowing a, b, and C, $\frac{A-B}{2}$ may be found, and $A + B$, or $180 - C$, is known; hence $\frac{A+B}{2}$ is known, and knowing $\frac{A+B}{2}$ and $\frac{A-B}{2}$, we may find A and B; since if $\frac{A+B}{2} = p$, and $\frac{A-B}{2} = q$, $A = p + q$, and $B = p - q$.

76. To find $\cos. \frac{A}{2}$, $\sin. \frac{A}{2}$, and $\tan. \frac{A}{2}$, in terms of the sides.

(1.) $\because \cos. A = \frac{b^2 + c^2 - a^2}{2bc}$;

$$\therefore 1 + \cos. A = \frac{b^2 + 2bc + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2bc};$$

$$\text{or } 2\cos.^2\frac{A}{2} = \frac{(b+c+a)(b+c-a)}{2bc}$$

$$= \frac{2S.(S-a)}{bc};$$

$$\therefore \cos.\frac{A}{2} = \sqrt{\frac{S.(S-a)}{bc}}.$$

$$(2.)\ 1 - \cos.A = \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} = \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{(a-b+c)(a+b-c)}{2bc};$$

$$\text{or } 2\sin.^2\frac{A}{2} = \frac{2(S-b).(S-c)}{bc};$$

$$\therefore \sin.\frac{A}{2} = \sqrt{\frac{(S-b).(S-c)}{bc}}.$$

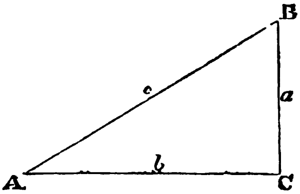
$$(3.)\ \therefore \tan.\frac{A}{2} = \frac{\sin.\frac{A}{2}}{\cos.\frac{A}{2}}$$

$$= \sqrt{\frac{(S-b).(S-c)}{S.(S-a)}}$$

These formulas, being adapted to logarithmic computation, are useful in practice.

Solution of Right-angled Triangles.

77. Let ABC be the triangle, C the right angle; let $A, B, C,$ denote the three angles of the triangle, and a, b, c, the sides opposite them, as before.



$$\text{Then } \because \ \sin.A = \frac{BC}{AB} = \frac{a}{c};$$

$$\cos.A = \frac{AC}{AB} = \frac{b}{c};$$

$$\tan.A = \frac{BC}{AC} = \frac{a}{b};$$

These formulas, with $c = \sqrt{a^2 + b^2}$, will solve any case of right-angled triangles.

To effect the solution, logarithmic tables are made use of; and since in these tables the logarithm of each of the trigonometrical ratios is increased by 10, we must, in practice, subtract 10 from the logarithm of the sine, cosine, &c.

Case 1. Given c, and A, find B, a, and b.

$$\text{Since } A + B = 90, \ \therefore \ B = 90 - A \text{ is known};$$

$$\text{also, } \sin.A = \frac{a}{c}, \text{ and } \cos.A = \frac{b}{c};$$

$$\therefore \ a = c \, . \, \sin.A, \text{ and } b = c \, . \, \cos.A;$$

$$\therefore \ \log.a = \log.c + \log.\sin.A - 10;$$
$$\log.b = \log.c + \log.\cos.A - 10.$$

Case 2. Given A and (a), find c, b, and B.

$$\angle \ B = (90 - A) \text{ is known};$$

$$\text{and } \sin.A = \frac{a}{c}, \text{ and } \tan.A = \frac{a}{b};$$

$$\therefore \ c = \frac{a}{\sin.A}, \text{ and } b = \frac{a}{\tan.A};$$

$$\therefore \ \log.c = \log.a - \log.\sin.A + 10;$$
$$\log.b = \log.a - \log.\tan.A + 10.$$

Case 3. Given A, and b, find B, a, and c.

$$B = 90 - A \text{ is found};$$

$$\tan. A = \frac{a}{b};\ \cos. A = \frac{b}{c};$$

$$\therefore\ a = b\ .\ \tan. A;\ \ c = \frac{b}{\cos. A};$$

$$\therefore\ \log. a = \log. b + \log. \tan. A - 10;$$
$$\log. c = \log. b - \log. \cos. A + 10.$$

Case 4. Given a and b, find A, B, and (c).

$$\text{Tan}. A = \frac{a}{b};$$

$$\therefore\ \log. \tan. A = 10 + \log. a - \log. b;\ \ \therefore\ A \text{ is known};$$
$$\text{and } B = 90 - A \text{ may be found};$$

$$\text{and } \sin. A = \frac{a}{c},\ \text{also } \cos. A = \frac{b}{c};$$

$$\therefore\ c = \frac{a}{\sin. A},\ \text{and } c = \frac{b}{\sin. B};$$

$$\therefore\ \log. c = 10 + \log. a - \log. \sin. A.$$
$$\text{or } \log. c = 10 + \log. b - \log. \sin. B.$$

Case 5. Given c and a, find A, B, and b.

$$\text{Here} \because \sin. A = \frac{a}{c};$$

$$\therefore\ \log. \sin. A = 10 + \log. a - \log. c,\ \text{whence } A;$$
$$\text{and } B = 90 - A;$$

$$\text{and } \cos. A = \frac{b}{c};\ \ \therefore\ b = c\ .\ \cos. A.$$

$$\therefore\ \log. b = \log. c + \log. \cos. A - 10.$$

But b may be found without knowing the angles.

For $b = \sqrt{c^2 - a^2} = \sqrt{(c+a).(c-a)}$;

$\therefore \log. b = \frac{1}{2}\{\log.(a+c) + \log.(c-a)\}$.

Oblique-angled Triangles.

78. The expressions obtained in the preceding pages, namely, $\cos. A = \dfrac{b^2 + c^2 - a^2}{2cb}$, and $\dfrac{\sin. A}{\sin. B} = \dfrac{a}{b}$, are those upon which the solution of oblique-angled triangles depends; each contains four unknown quantities, of which three being given, the fourth may be found.

79. Given the three sides of a triangle, to find the angles.

Let A, B, C, be the angles,
a, b, c, the sides opposite.

Then, $\cos. A = \dfrac{b^2 + c^2 - a^2}{2bc}$, whence $\cos. A$, and A; but this formula is not adapted to logarithmic computation; but $\cos. \dfrac{A}{2}$, $\sin. \dfrac{A}{2}$, and $\tan. \dfrac{A}{2}$ are; we may therefore make use of either of their values. Now, (see Art. 76,)

$$\cos. \frac{A}{2} = \sqrt{\frac{S.(S-a)}{bc}};$$

$$\therefore \log. \cos. \frac{A}{2} = 10 + \tfrac{1}{2}\{\log. S + \log.(S-a) - \log. b - \log. c\}.$$

And, by substitution,

$$\log. \cos. \frac{B}{2} = 10 + \tfrac{1}{2}\{\log. S + \log.(S-b) - \log. a - \log. c\};$$

and

$$\log. \cos. \frac{C}{2} = 10 + \tfrac{1}{2}\{\log. S + \log.(S-c) - \log. a - \log. b\}.$$

Second Method.

$$\sin.\frac{A}{2} = \sqrt{\frac{(S-b).(S-c)}{bc}}.$$

$$\therefore \log.\sin.\frac{A}{2} = 10 + \tfrac{1}{2}\{\log.(S-b) + \log.(S-c) - \log.b - \log.c\}.$$

Similarly,

$$\log.\sin.\frac{B}{2} = 10 + \tfrac{1}{2}\{\log.(S-a) + \log.(S-c) - \log.a - \log.c\}.$$

$$\log.\sin.\frac{C}{2} = 10 + \tfrac{1}{2}\{\log.(S-a) + \log.(S-b) - \log.a - \log.b\}.$$

Third Method.

$$\tan.\frac{A}{2} = \sqrt{\frac{(S-b).(S-c)}{S.(S-a)}};$$

Therefore,

$$\log.\tan.\frac{A}{2} = 10 + \tfrac{1}{2}\{\log.(S-b) + \log.(S-c) - \log.S - \log.(S-a)\}.$$

Fourth Method.

Since $\sin.A = \frac{2}{bc}\sqrt{S.(S-a).(S-b).(S-c)}$,

Therefore,

$$\log.\sin.A = 10 + \log.2 - \log.b - \log.c + \tfrac{1}{2}\{\log.S + \log.(S-a) + \log.(S-b) + \log.(S-c)\}.$$

By any of these methods may the angles be found; the last method, however, must not be used, if the angle be near 90°, since the logarithms of the sines of arcs near 90° differ but little. If all the angles be required, we may use the third method, since the same factors are made use of in every computation.

80. Given a, and two angles A and B; find C, b, and c.

$C = 180 - (A + B)$ is therefore known.

$$\text{Also } \frac{b}{a} = \frac{\sin. B}{\sin. A}, \text{ and } \frac{c}{a} = \frac{\sin. C}{\sin. A};$$

$$\therefore\ b = a \,.\, \frac{\sin. B}{\sin. A}, \text{ and } c = a \,.\, \frac{\sin. C}{\sin. A};$$

$$\therefore\ \log. b = \log. a + \log. \sin. B - \log. \sin. A,$$
$$\text{and } \log. c = \log. a + \log. \sin. C - \log. \sin. A.$$

81. Given two sides a and b, and A, find B, C, and c.

$$\frac{\text{Sin}. B}{\text{Sin}. A} = \frac{b}{a};$$

$$\therefore\ \sin. B = \frac{b}{a} \sin. A;$$

$$\therefore\ \log. \sin. B = \log. b + \log. \sin. A - \log. a;$$

$$\therefore\ B \text{ is known, and therefore } C = 180 - (A + B),$$

$$\text{and } c = a \,.\, \frac{\sin. C}{\sin. A};$$

$$\therefore\ \log. c = \log. a + \log. \sin. C - \log. \sin. A.$$

If A be an acute angle, and $a \angle b$, we shall have two solutions; for B and $(180 - B)$ having the same sine, we shall be unable to determine which of the two is the correct angle.

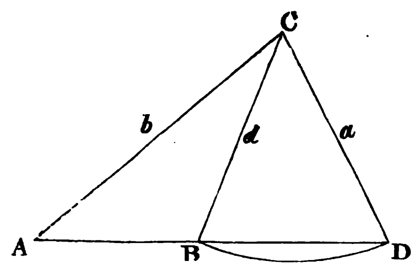
This is termed the *ambiguous case.*

If, however, A be an obtuse angle, then B must be an acute angle, and there is no ambiguity.

Also, if A be acute, and $a \; 7 \; b$, then, since the greater side is opposite to the greater angle, $A \; 7 \; B$, and therefore B must also be an acute angle; this case is therefore free from ambiguity.

This ambiguous case has already been alluded to; it may, however, be further illustrated by the following explanation, and by the use of the annexed figure.

Let A be the given angle, $CA = b$ one of the given sides, and $a \angle b$. With centre C and radius $CB = a$, describe an arc cutting AB produced in D.



Then $\because \angle CBD = \angle CDB$; $\therefore CBA$, which is the supplement of CBD, is also the supplement of CDB.

If, therefore, we assume the required angle B to be acute, we must take the triangle CDA, which contains the given quantities, a, b, and $\angle A$.

If we suppose the required angle to be obtuse, we must take the triangle ABC, which contains a, b, and $\angle A$, and also $\angle CBA = 180 - \angle CDA = \overline{180 - B}$.

82. Two sides a, b, and included $\angle C$ being given, find c, and A, and B.

$$\text{Since } \frac{a+b}{a-b} = \frac{\tan. \frac{A+B}{2}}{\tan. \frac{A-B}{2}};$$

$$\therefore \tan. \frac{A-B}{2} = \frac{a-b}{a+b} \tan. \left(\frac{A+B}{2}\right) = \frac{a-b}{a+b} \tan. \left(90 - \frac{C}{2}\right)$$

$$= \frac{a-b}{a+b} \cot. \frac{C}{2};$$

$$\therefore \log.\tan.\frac{A-B}{2} = \log.(a-b) + \log.\cot.\frac{C}{2} - \log.(a+b);$$

whence $\frac{A-B}{2}$ may be computed.

$$\text{Let } \frac{A-B}{2} = m.$$

$$\text{Also } \frac{A+B}{2} = \frac{180-C}{2} = 90 - \frac{C}{2};$$

$$\therefore A = 90 + m - \frac{C}{2}, \text{ and } B = 90 - m - \frac{C}{2}.$$

$$\text{And } c = a\frac{\sin. C}{\sin. A};$$

$$\therefore \log. c = \log. a + \log.\sin. C - \log.\sin. A.$$

83. The value of c may be found without the computation of A and B, for $c = \sqrt{a^2 + b^2 - 2ab.\cos. C}$; but this formula is not adapted to logarithmic computation, but it may be easily put under the proper form.

$$\text{For } \because \cos. C = 2\cos.^2\frac{C}{2} - 1;$$

$$\begin{aligned} \therefore c^2 &= a^2 + b^2 - 4ab.\cos.^2\frac{C}{2} + 2ab \\ &= (a+b)^2 - 4ab.\cos.^2\frac{C}{2} \\ &= (a+b)^2\left\{1 - \frac{4ab.\cos.^2\frac{C}{2}}{(a+b)^2}\right\}. \end{aligned}$$

$$\text{Let } \frac{4ab.\cos.^2\frac{C}{2}}{(a+b)^2} = \sin.^2\theta.$$

$$\therefore\ c^2 = (a+b)^2\{1 - \sin.^2\theta\} = (a+b)^2\cos.^2\theta;$$

$$\therefore\ c = (a+b)\cos.\theta;$$

$$\therefore\ \log. c = \log.(a+b) + \log.\cos.\theta - 10.$$

Also, since $\sin.\theta = \dfrac{2\cos.\frac{C}{2}\sqrt{ab}}{a+b}$;

$$\therefore \log.\sin.\theta = \log.2 + \log.\cos.\frac{C}{2} + \tfrac{1}{2}\{\log.a + \log.b\} - \log.(a+b),$$

whence $\sin.\theta$, and $\therefore\ \theta$, and c may be found.

Or, since $\cos. C = 1 - 2\sin.^2\frac{C}{2}$,

$$c^2 = a^2 + b^2 - 2ab + 4ab.\sin.^2\frac{C}{2}$$

$$= (a-b)^2 + 4ab.\sin.^2\frac{C}{2}$$

$$= \overline{a-b}|^2\left\{1 + \frac{4ab.\sin.^2\frac{C}{2}}{\overline{a-b}|^2}\right\}$$

$$= (a-b)^2\{1 + \tan.^2\theta\};\ \text{if } \tan.^2\theta = \frac{4ab.\sin.^2\frac{C}{2}}{\overline{a-b}|^2}$$

$$= (a-b)^2\sec.^2\theta;$$

$$\therefore\ c = (a-b)\sec.\theta,$$

$$\text{and } \log. c = \log.(a-b) + \log.\sec.\theta - 10.$$

Also, $\tan.\theta = \dfrac{2\sin.\frac{C}{2}\sqrt{ab}}{(a-b)}$;

$$\therefore \log.\tan.\theta = \log.2 + \log.\sin.\frac{C}{2} + \tfrac{1}{2}(\log.a + \log.b) - \log.(a-b);$$

by either of these formulas may (c) be found.

COR. Since $\tan.\dfrac{A-B}{2}=\dfrac{a-b}{a+b}\tan.\dfrac{A+B}{2}$;

$$\text{let } a = b\,.\,\tan.\,\theta, \text{ or } \frac{a}{b} = \tan.\,\theta;$$

$$\therefore\ \frac{a-b}{a+b}=\frac{\tan.\,\theta-1}{\tan.\,\theta+1}=\tan.\,(\theta-45);$$

$$\therefore\ \tan.\frac{A-B}{2}=\tan.\,(\theta-45)\tan.\frac{A+B}{2}$$

$$=\tan.\,(\theta-45)\cot.\frac{C}{2};$$

$$\therefore\ \log.\tan.\frac{A-B}{2}=\log.\tan.(\theta-45)+\log.\cot.\frac{C}{2}-10.$$

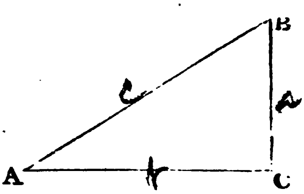
And $\tan.\,\theta$ may be computed from the formula,

$$\log.\,\tan.\,\theta = \log.\,a - \log.\,b + 10;$$

whence θ may be conveniently found, and therefore $\dfrac{A-B}{2}$ may be computed from a table of tangents only.

84. The use of the formulas may be shown by the actual solution of a few cases of the triangles.

Ex. 1. In the right-angled triangle ABC,
Given $b = 85.6$ feet,
$A = 63^\circ\,25'$,
find a.



$$\text{Tan.}\,A=\frac{a}{b};\ \therefore\ a = b\,.\,\tan.\,A;$$

$$\text{whence } \log.\,a = \log.\,b + \log.\,\tan.\,A - 10.$$

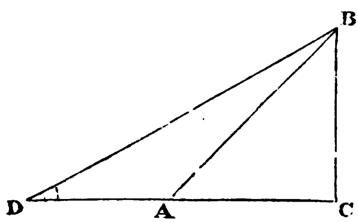
$$\begin{aligned} \text{Now } \log.\,b &= 1.9324738 \\ \log.\,\tan.\,A - 10 &= \ \ .3006836 \\ \hline & \ \ 2.2331574 = \log.\,171.063; \end{aligned}$$

$$\therefore\ a = BC = 171.063 \text{ feet.}$$

This problem gives the height of a tower whose base is accessible. The distance CA is measured, and $\angle\, BAC$ is observed.

Observe, the $\angle\, BAC$ is called the angle of elevation; also if a line BD be drawn from B parallel to CA, the $\angle\, DBA$, is called the $\angle$ of depression of the point A.

Problem 1. Find the height of an inaccessible object standing upon the horizontal plane.



A, a place from which B is visible, observe the angle BAC; measure AD, (D being a convenient place for observing B, and AD in the same direction as CA,) and observe the angle BDC.

Then,

$$BC = AB \,.\, \text{sin.} A. \text{ But } \frac{AB}{AD} = \frac{\text{sin.} D}{\text{sin.} ABD} = \frac{\text{sin.} D}{\text{sin.}(A-D)};$$

$$\therefore AB = \frac{AD \,.\, \text{sin.} D}{\text{sin.}(A-D)}, \text{ and } BC = \frac{AD \,.\, \text{sin.} D \,.\, \text{sin.} A}{\text{sin.}(A-D)}.$$

Ex. 2. Let $\angle\, A = 60^\circ$, and $AD = 100$ feet.

$$\angle\, D = 45^\circ; \ \therefore A - D = 15^\circ;$$

$$\therefore BC = \frac{100 \,.\, \text{sin.} 45 \,.\, \text{sin.} 60}{\text{sin.} 15}.$$

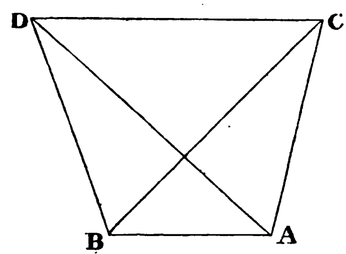
Therefore,

$$\log. BC = \log. 100 + \log. \text{sin.} 60 + \log. \text{sin.} 45 - (\log. \text{sin.} 15 + 10);$$

$$
\begin{aligned}
\text{Log. } 100 &= \ \ 2.0000000 \\
\text{Log. sin. } 60 &= \ \ 9.9375306 \\
\text{Log. sin. } 45 &= \ \ \underline{9.8494850} \\
& \ \ \ 21.7870156 \\
(\text{Log. sin. } 15 + 10) &= \underline{19.4129962} \\
& \ \ \ \ 2.3740194 = \text{log. } 236.601;
\end{aligned}
$$

$$\therefore \ BC = 236.6 \text{ feet.}$$

Problem 2. Two inaccessible objects, D and C, are both visible from each of the two places A and B, whose distance is known, and each of which is visible from the other. Required the distance CD.



Observe the $\angle\ CAD = \alpha$, and $\angle\ DAB = \beta$ from A;

and $\angle\ DBC = \alpha_{,}$, and $\angle\ CBA = \beta_{,}$ from B.

Let $AB = a$.

Then, if we can find DA and AC, or DB and BC, the problem is reduced to this; find the third side, when the two other sides, and the angle included by them, are given.

Now in $\Delta\ CAB$, $\dfrac{CA}{AB} = \dfrac{\text{sin. } CBA}{\text{sin. } ACB} = \dfrac{\text{sin. } \beta_{,}}{\text{sin. } ACB}$.

But

$$ACB = 180 - (ABC + CAB) = 180 - (\beta_{,} + \alpha + \beta);$$

$$\therefore \ \text{sin. } ACB = \text{sin. } (\alpha + \beta + \beta_{,});$$

$$\therefore \ CA = \frac{a \ . \ \text{sin. } \beta_{,}}{\text{sin. } (\alpha + \beta + \beta_{,})}.$$

And in $\Delta\ DBA$, $\dfrac{DA}{AB} = \dfrac{\sin. ABD}{\sin. ADB} = \dfrac{\sin.(\alpha_{,} + \beta_{,,})}{\sin. ADB}$;

$$ADB = 180 - (ABD + BAD) = 180 - (\alpha_{,} + \beta_{,} + \beta);$$

$$\therefore \sin. ADB = \sin.(\beta + \beta_{,} + \alpha_{,});$$

$$\therefore AD = a \cdot \frac{\sin.(\alpha_{,} + \beta_{,,})}{\sin.(\beta + \beta_{,} + \alpha_{,})}.$$

And thus knowing AC and AD, and $\angle\ CAD$, we may, by means of a subsidiary angle, compute CD.

Ex. 3. The sides of a triangle are 425, 681, and 324 feet; find the angle opposite to the greatest side.

Let a be the greatest side, and therefore A the greatest angle.

Now $\log. \cos. \dfrac{A}{2} = 10 + \frac{1}{2}\{\log. S + \log.(S - a) - \log. b - \log. c\}$

$a = 681$	$\log. S = 2.8543060$	$\log. b = 2.6283889$
$b = 425$	$\log.(S-a) = 1.5314789$	$\log. c = 2.5105450$
$c = 324$	$2)4.3857849$	$2)5.1389339$
$2S = 1430$	2.1928924	2.5694669
$S = 715$		2.1928924
$S - a = 34$		$.3765745$

$$\therefore \log. \cos. \frac{A}{2} = 10 - .3765745 = 9.6234255.$$

$$\text{Whence } \frac{A}{2} = 65\ 9'\ 16'', \text{ and } A = 130^\circ\ 18'\ 32''.$$

B may be found from $\sin. B = \dfrac{b}{a} \cdot \sin. A$, and $C = 180 - (A + B)$.

Ex. 4. To find the area of the same triangle.

$$\text{Area} = \sqrt{S \cdot (S-a) \cdot (S-b) \cdot (S-c)}.$$

$$\log. \text{area} = \tfrac{1}{2}\{\log. S + \log.(S-a) + \log.(S-b) + \log.(S-c)\}.$$

$$S = 715,\ S - a = 34;\ S - b = 290,\ S - c = 391,$$

$$\begin{aligned}
\log. S &= 2.8543060\\
\log.(S-a) &= 1.5314789\\
\log.(S-b) &= 2.4623980\\
\log.(S-c) &= 2.5921768\\
2\,)\,&\overline{9.4403597}\\
&\underline{4.7201798} = \log. 52502.5;
\end{aligned}$$

$\therefore$ area $= 52502.5$ square feet.

Ex. 5. Two sides of a triangle are 891.6 and 732.8 feet, and the angle opposite the greater is $72^\circ\ 8'$; find the angle opposite to the less.

A the greater and B the less angle;

$$\therefore \sin. B = \frac{b}{a} . \sin. A;$$

$$\begin{aligned}
\log. \sin. B &= \log. b - \log. a + \log. \sin. A\\
&= \log. \sin. A - (\log. a - \log. b);
\end{aligned}$$

$a = 891.6$	$A = 72^\circ\ 8'$.
$b = 732.8$	
$\log. a = 2.9501701$	$\log. \sin. A = 9.9785334$
$\log. b = 2.8649855$	$.0851846$
$.0851846$	$\therefore \log. \sin. B = 9.8933488$

$$\therefore B = 51^\circ\ 28'\ 3''.$$

Ex. 6. Two sides of a triangle are 85.63 and 78.21 feet, they include the angle $48^\circ\ 24'$. Find the remaining sides and angles.

$a = 85.63$	$C = 48^\circ\ 24$
$b = 78.21$	$\therefore A + B = 131^\circ\ 36'$.

$$a + b = 163.84 \qquad \frac{C}{2} = 24^\circ\ 12'$$
$$a - b = 7.42 \qquad \frac{A+B}{2} = 65^\circ\ 48.$$

Now $\log.\tan.\left(\frac{A-B}{2}\right) = \log.\cot.\frac{C}{2} + \log.(a-b) - \log.(a+b)$;

$$\begin{aligned}
\log.\cot.\frac{C}{2} &= 10.3473497 \\
\log.(a-b) &= .8704039 \\
& \overline{11.2177536} \\
\log.(a+b) &= 2.2144199 \\
& \overline{9.0033337} = \log.\tan. 5^\circ\,45'\,15'';
\end{aligned}$$

$$\therefore \frac{A-B}{2} = 5^\circ\,45'\,15'';$$
$$\text{and } \frac{A+B}{2} = 65^\circ\,48';$$
$$\therefore A = 71^\circ\,33'\,15'', \text{ and } B = 60\ \ 2'\,45'';$$

$$\text{and } c = a \,.\, \frac{\sin. C}{\sin. A};$$

$\therefore \log. c = \log. a + \log.\sin. C - \log.\sin. A,$

Whence (c) may be found $= 67.502$.

But to compute (c) independently of A and B.

By Art. 83 we have,

$$\log. c = \log.(a+b) + \log.\cos.\theta - 10;$$

$$\log.\sin.\theta = \log. 2 + \log.\cos.\frac{C}{2} + \tfrac{1}{2}\{\log. a + \log. b\} - \log.(ab+\ );$$

$$\begin{array}{r|l}
\log. a = 1.9326259 & \log. 2 = .3010300 \\
\log. b = \underline{1.8932623} & \log. \cos. \frac{C}{2} = 9.9600520 \\
2)\underline{3.8258882} & \phantom{\log. \cos. \frac{C}{2} = }\underline{1.9129441} \\
\underline{\underline{1.9129441}} & \phantom{\log. \cos. \frac{C}{2} = }12.1740261 \\
 & \log.(a+b) = \underline{2.2144199} \\
 & \log. \sin. \theta = \underline{\underline{9.9596062}} = \log. \sin. 65^\circ 40' 10'';
\end{array}$$

$$\therefore \theta = 65^\circ\ 40'\ 10''.$$

$$\begin{aligned}
\log. (a + b) &= 2.2144199 \\
\log. \cos. \theta &= \underline{9.6148976} \\
& 11.8293175 \\
& \underline{10} \\
\therefore \log. c &= \underline{\underline{1.8293175}} = \log. 67.5021;
\end{aligned}$$

$$\text{or } c = 67.5021 \text{ feet.}$$

Ex. 7. Find the area of the same Δ.

$$\because \text{ area} = \frac{ab \,.\, \sin. C}{2} \ldots . \text{ (Art. 73.)}$$

$$\therefore \log. \text{area} = \log. a + \log. b + \log. \sin. C - (10 + \log. 2).$$

$$\begin{aligned}
\log. a &= 1.9326259 \\
\log. b &= 1.8932623 \\
\log. \sin. C &= \underline{9.8737844} \\
& 13.6996726 \\
10 + \log. 2 &= \underline{10.3010300} \\
\therefore \log. \text{area} &= \underline{\underline{3.3986426}} = \log. 2504.047;
\end{aligned}$$

$$\therefore \text{area} = 2504.047 \text{ square feet.}$$

EXAMPLES.

(1.) Right-angled triangles; $C = 90$.

$$\left.\begin{aligned} \text{Given } b &= 469.34 \\ A &= 51^\circ 26' 17'' \end{aligned}\right\} \therefore \begin{aligned} a &= 588.7 \\ c &= 752.9 \end{aligned}$$

$$\left.\begin{aligned} \text{Given } c &= 4184 \\ b &= 2632 \end{aligned}\right\} \therefore \begin{aligned} a &= 3252.4 \\ A &= 51^\circ 1' 8'' \end{aligned}$$

$$\left.\begin{aligned} \text{Given } a &= 851 \\ b &= 763 \end{aligned}\right\} \therefore \begin{aligned} A &= 48^\circ 7' 16'' \\ c &= 1142.9 \end{aligned}$$

(2.) Oblique-angled triangles.

$$\left.\begin{aligned} \text{Given } a &= 785.8 \\ b &= 720.8 \\ c &= 643.2 \end{aligned}\right\} \therefore \begin{aligned} A &= 70^\circ 5' 22'' \\ C &= 50^\circ 19' 11'' \end{aligned}$$

$$\left.\begin{aligned} \text{Given } b &= 720.8 \\ A &= 70^\circ 5' 22'' \\ B &= 59^\circ 35' 27'' \end{aligned}\right\} \therefore \begin{aligned} a &= 785.8 \\ c &= 643.2 \end{aligned}$$

$$\left.\begin{aligned} \text{Given } c &= 697 \\ A &= 81^\circ 30' 10'' \\ C &= 40^\circ 30' 44'' \end{aligned}\right\} \therefore \begin{aligned} a &= 813. \\ b &= 534. \end{aligned}$$

$$\left.\begin{aligned} \text{Given } a &= 874.56 \\ b &= 859.56 \\ C &= 91^\circ 58' 10'' \end{aligned}\right\} \therefore \begin{aligned} A &= 44^\circ 29' 39''.5 \\ B &= 43^\circ 32' 10''.5 \\ c &= 1247.14 \end{aligned}$$

It is obvious, by changing the given quantities into required ones, that from these examples we may make very numerous illustrations of the formulas.

85. To these numerical results we will add a few questions, illustrative of the formulas contained in the preceding pages.

Prob. (1.) The angles, A, B, C, of the triangle ABC are as the numbers 2, 3, 4, respectively; show that

$$2 \cos. \frac{A}{2} = \frac{a + c}{b}.$$

It is obvious that $B = \frac{3A}{2}$, and that $C = 2A$.

Now, $\sin. A + \sin. C = 2 \sin. \frac{A + C}{2} . \cos. \frac{C - A}{2}$.

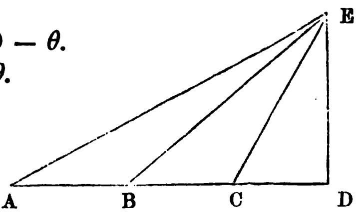
$$\text{But } \frac{A + C}{2} = \frac{3A}{2} = B, \text{ and } \frac{C - A}{2} = \frac{A}{2};$$

$$\therefore\ 2 \sin. B . \cos. \frac{A}{2} = \sin. C + \sin. A;$$

$$\therefore\ 2 \cos. \frac{A}{2} = \frac{\sin. C}{\sin. B} + \frac{\sin. A}{\sin. B} = \frac{c}{b} + \frac{a}{b} = \frac{c + a}{b}.$$

(2.) The elevation of a tower is observed. At a station (a) feet nearer the elevation is the complement of the former; (b) feet nearer still, it is double the first elevation. Show that the height $= \sqrt{(a + b)^2 - \frac{a^2}{4}}$.

$AB = a$. $\angle EAD = \theta$.
$BC = b$. $\angle EBD = 90 - \theta$.
$CD = x$. $\angle ECD = 2\theta$.
$ED = h$.



Then $\angle CEA = \angle CAE$; $\therefore\ CE = CA = (a + b)$.

And $h^2 = (a + b)^2 - x^2 = (a + b + x) . (a + b - x)$.

But $\tan. \theta = \frac{h}{a + b + x}$, and $\tan. (90 - \theta) = \cot. \theta = \frac{h}{b + x}$;

$$\therefore \frac{h}{a+b+x} = \frac{b+x}{h};$$

$$\therefore h^2 = (a+b+x)\,.\,(b+x) = (a+b+x)\,.\,(a+b-x);$$

$$\therefore b+x = a+b-x; \;\; \therefore 2x = a, \text{ and } x = \frac{a}{2};$$

$$\therefore h = \sqrt{(a+b)^2 - \frac{a^2}{4}}.$$

(3.) Given the perimeter of a triangle and the ratios of its angles, find the sides.

Let P be the given perimeter;

A, B, C, the angles, and a, b, c, the sides.

Let $B = mA$, $C = nA$; $\therefore A + B + C = \pi = (1+m+n)A$;

$$\therefore A = \frac{\pi}{1+m+n};\; B = \frac{m\pi}{1+m+n};\; C = \frac{n\pi}{1+m+n},$$

and the angles are known.

$$\text{But } P = a+b+c = a\left\{1 + \frac{b}{a} + \frac{c}{a}\right\} = a\left\{1 + \frac{\sin. B}{\sin. A} + \frac{\sin. C}{\sin. A}\right\};$$

$$\therefore a = \frac{P \sin. A}{\sin. A + \sin. B + \sin. C}$$

$$= \frac{2P \sin. \frac{A}{2}\,.\,\cos. \frac{A}{2}}{4 \cos. \frac{A}{2}\,.\,\cos. \frac{B}{2}\,.\,\cos. \frac{C}{2}}$$

$$= \frac{P\,.\,\sin. \frac{A}{2}}{2 \cos. \frac{B}{2}\,.\,\cos. \frac{C}{2}};$$

whence b and c may also be found.

(4.) The sides of a triangle are three consecutive numbers, and the greatest angle is double the least; determine the triangle.

Let A be the greatest and C the least angle; and let $(x-1)a$, xa, $(x+1).a$, be the three sides, c, b, a.

$$\text{Then } \frac{\sin. A}{\sin. C} \text{ or } \frac{\sin. 2C}{\sin. C} = 2\cos. C = \frac{x+1}{x-1}.$$

$$\text{But } 2\cos. C = \frac{(x+1)^2 + x^2 - (x-1)^2}{x.(x+1)} = \frac{x^2+4x}{x.(x+1)} = \frac{x+4}{x+1};$$

$$\therefore \frac{x+1}{x-1} = \frac{x+4}{x+1}; \therefore x^2+2x+1 = x^2+3x-4; \therefore x = 5;$$

and the sides are as the numbers 4, 5, 6; the factor (a) may be of any length, a foot, a yard, a mile.

(5.) Three sides of a quadrilateral inscribed in a circle being given, find the fourth, it being the diameter of the circle.

Let a, b, c, be the given sides,

$2A$, $2B$, $2C$, the angles they subtend at the centre,

x = the fourth side = $2r$, r being radius of circle.

Then $2A + 2B + 2C = 180$;

$\therefore A + B + C = 90$, and $C = 90 - (A + B)$.

But $a = 2r \sin. A = x \sin. A$; and $b = x \sin. B$;

$$c = x\sin. C = x\cos.(A+B) = x(\cos. A.\cos. B - \sin. A.\sin. B);$$

$$\therefore \frac{c}{x} = \sqrt{1-\frac{a^2}{x^2}}.\sqrt{1-\frac{b^2}{x^2}} - \frac{ab}{x^2};$$

$$\therefore \frac{c}{x} + \frac{ab}{x^2} = \sqrt{1 - \frac{a^2+b^2}{x^2} + \frac{a^2b^2}{x^4}};$$

$$\therefore \frac{c^2}{x^2} + \frac{2abc}{x^3} + \frac{a^2b^2}{x^4} = 1 - \frac{a^2+b^2}{x^2} + \frac{a^2b^2}{x^4};$$

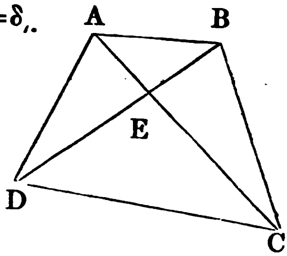
$$\therefore x^3 - (a^2+b^2+c^2)x - 2abc = 0,$$

a cubic equation having but one real root.

If $a = b = c$, $x = 2a$, which is obvious, since the figure becomes the half of a regular hexagon.

(6.) Given the two diagonals of a quadrilateral field, and the angle at which they intersect; find its area.

Let $\angle AED = \theta$, $AC = \delta$, $BD = \delta_{,}$.



Then area of

$$\Delta ADB = \delta_{,} . \frac{AE \sin.\theta}{2};$$

$$\Delta BCD = \delta_{,} . \frac{CE \times \sin.\theta}{2};$$

$\therefore$ quadrilateral

$$= \frac{\delta_{,} . \sin.\theta}{2} . (AE + CE) = \frac{\delta . \delta_{,} . \sin.\theta}{2}.$$

(7.) Three places, A, B, C, being given in the map of a country, to determine the position of a fourth point D, the angles BDA, BDC, being known.

$AB = a$. $\angle BDA = \alpha$. $\angle BAD = \theta$. $BD = x$.
$BC = b$. $\angle BDC = \beta$. $\angle BCD = \phi$.
$AC = c$.

Then $\angle ABC$ may be found from the sides a, b, c.

$$\text{And} \because \angle B + \angle D + \angle A + \angle C = 2\pi;$$

$$\therefore B + \alpha + \beta + \theta + \phi = 2\pi; \therefore \phi = m - \theta;$$

$$\text{where } m = 2\pi - (B + \alpha + \beta).$$

$$\text{But } \frac{x}{a} = \frac{\sin.\theta}{\sin.\alpha} (1); \text{ and } \frac{x}{b} = \frac{\sin.\phi}{\sin.\beta} (2);$$

$$\therefore \frac{a . \sin.\beta}{b . \sin.\alpha} = \frac{\sin.\phi}{\sin.\theta} = \frac{\sin.\overline{m - \theta}}{\sin.\theta} = \frac{\sin.m . \cos.\theta - \sin.\theta . \cos.m}{\sin.\theta};$$

$$\therefore \sin. m . \cot. \theta - \cos. m = \frac{a}{b} . \frac{\sin. \beta}{\sin. a};$$

$$\therefore \cot. \theta = \frac{\cos. m}{\sin. m} + \frac{a . \sin. \beta}{b . \sin. a . \sin. m} = \frac{\cos. m}{\sin. m} + \frac{\cos. n}{\sin. n},$$

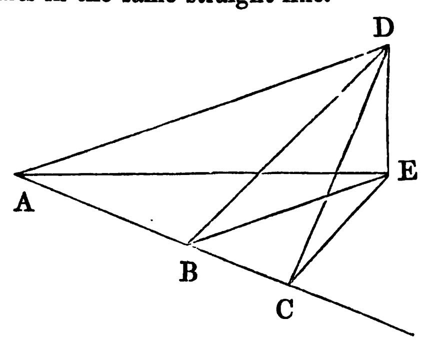
$$\text{by making } \cot. n = \frac{a . \sin. \beta}{b . \sin. a . \sin. m};$$

$$\therefore \cot. \theta = \frac{\sin. (m + n)}{\sin. m . \sin. n}, \text{ whence } \theta \text{ may be found.}$$

And (x) or BD can be found from equation (1).

Also, in the $\Delta\, ABD$, AB being given, BD and $\angle\, ABD$ being computed, AD may be found; similarly, the value of DC may be obtained, and thus the position of D determined.

(8.) Find the altitude of an object above a horizontal plane, by means of three observations taken from three given points in the same straight line.



D the object.
A, B, C, the places of observation.
DE the altitude required $= h$.
$AB = a$, $\angle\, DAE = a$.
$BC = b$, $\angle\, DBE = \beta$.
$\angle\, DCE = \gamma$.

Then $AE = h \cdot \cot. \alpha$, $BE = h \cdot \cot. \beta$, $CE = h \cdot \cot. \gamma$.

$$\text{But } \cos. ABE = \frac{AB^2 + BE^2 - AE^2}{2AB \cdot BE};$$

$$\cos. EBC = -\cos. ABE = \frac{EB^2 + BC^2 - CE^2}{2EB \cdot BC};$$

$$\therefore \frac{AB^2 + BE^2 - AE^2}{AB} = -\frac{EB^2 + BC^2 - CE^2}{BC};$$

$$b \cdot (a^2 + h^2 \cdot \cot.^2\beta - h^2 \cdot \cot.^2\alpha) = -a(h^2 \cdot \cot.^2\beta + b^2 - h^2 \cdot \cot.^2\gamma);$$

$$h^2 \{a \cdot \cot.^2\gamma + b \cdot \cot.^2\alpha - (a+b)\cot.^2\beta\} = a^2 b + a b^2;$$

$$\therefore h = \sqrt{\frac{(a+b)ab}{a \cdot \cot.^2\gamma + b \cdot \cot.^2\alpha - (a+b)\cot.^2\beta}}.$$

Cor. If $b = a$, or B be equidistant from A and C.

$$h = a\sqrt{\frac{2}{\cot.^2\alpha + \cot.^2\gamma - 2\cot.^2\beta}}.$$

EXAMPLES.

1. In any triangle of which the angles are A, B, C, and sides a, b, c, show that

(1.) $c = \dfrac{b}{\cos. A + \sin. A \cdot \cot. C} = a \cdot (\cos. B + \sin. B \cdot \cot. A)$

$= b \cdot \cos. A \pm \sqrt{a^2 - b^2 \cdot \sin.^2 A} = a \cos. B + b \cos. A.$

(2.) $\text{Sin.} A = \dfrac{\sin. C \{b \cdot \cos. C \pm \sqrt{c^2 - b^2 \cdot \sin.^2 C}\}}{c}.$

(3.) Also, $\cot. \dfrac{A}{2} : \cot. \dfrac{B}{2} :: b + c - a : a + c - b.$

(4.) .... $\sin. (A - B) : \sin. C :: a^2 - b^2 : c^2.$

2. If $2\cos. B \cdot \sin. C = \sin. A$; the triangle is isoscele.

3. Find the area of a parallelogram, two of its sides being 425 and 320 feet, and smaller $\angle = 18^\circ 12'$.

Area = 42477.54 feet.

4. Each side of a rhombus is 47 yards, and one angle is 45°; find its area.

Area = 1562 yards.

5. The diagonals of a parallelogram are 40 and 36 feet, and the angle of intersection is 30°; find area.

Area = 40 yards.

6. The side of an equilateral four-sided figure $= a$, and one of the angles is 45°; show that the two diagonals are respectively, $a\sqrt{2+\sqrt{2}}$ and $a\sqrt{2-\sqrt{2}}$.

7. Find the area of a square, when the difference between the diagonal and the side $= m$ feet.

8. Given the hypothenuse of a right-angled triangle, and the sum of the other two sides; find the sides and angles.

Ex. Let $c = 500$; $a + b = 700$.

9. The area of an isosceles triangle $= m^2$; the base $= 2c$; find the angle B at the base.

$$B = \tan.^{-1}\left(\frac{m^2}{c^2}\right).$$

10. In the triangle ABC, take a point D, join DA, DB, DC. Given AB, AC, $\angle ABD$, $\angle ACD$, and $\angle BDC$; find BC.

11. Given c, the base of an isosceles triangle, and p, the perpendicular from one of the equal angles upon the opposite side; show that area $= \dfrac{pc^2}{4\sqrt{c^2-p^2}}$.

12. The perimeter of a right-angled triangle is twelve feet, the area six square feet; find the sides.

13. The angles of a triangle are as the numbers 1, 2, 3, its perimeter is 20; show that the greatest side $= 20\left(1 - \frac{1}{\sqrt{3}}\right)$.

14. If the angles be in geometrical proportion, ratio $\frac{1}{2}$, the greatest side $=$ 2 perimeter $\times$ sin. $12^\circ\ 51'\ 25''\ \frac{5}{7}$.

15. The sides of a triangle are in arithmetical proportion, its area $= \frac{3}{5}$ of the area of an equilateral triangle, having the same perimeter; show that the greatest angle $= 120$, and the sides are as $3 : 5 : 7$.

16. The angles of a triangle are as the numbers 1, 2, 3; and the perpendicular from the greatest angle upon the opposite is (p). Show that the area $= \frac{2p^2}{\sqrt{3}}$.

17. Given in a right-angled triangle, the lines drawn from the acute angles to the bisections of the opposite sides; find the angles.

18. From the top of a hill there are observed two consecutive milestones, on a horizontal road, running directly from the base. The angles of depression are found to be 45° and 30°; find the height of the hill.

19. If θ be the angle between the diagonals of a parallelogram whose sides a, b, are inclined at an angle A to each other; then tan. $\theta = \frac{2ab \sin. A}{a^2 - b^2}$.

20. The angles of a triangle are as the numbers 2, 3, and 5, and the side opposite to the greatest angle $= 100$ feet; find the remaining sides and the angles.

21. If CD bisect AB in the triangle ABC,

$$CD = \tfrac{1}{2}\sqrt{2(a^2 + b^2) - c^2}.$$

22. Prove that the area of a triangle

$$= \frac{a^2 + b^2 + c^2}{4(\text{cot.}\,A + \text{cot.}\,B + \text{cot.}\,C)}.$$

23. Find a point within a triangle from which the three sides shall subtend equal angles.

24. Given the three perpendiculars from the angles on the opposite sides; find the sides.

25. Given the perimeter and the area of a triangle and (A); find a.

26. Given the area, (C) and $a + b$; find the sides.

27. Given c, C and $a + b$; find a and b.

28. Given the area, side c, and $A + B$; find $A - B$.

29. If L = length in miles of an arc of a great circle of the earth, D the depression in feet of one extremity of it, below a tangent to the other, $D = \frac{2}{3} \cdot L^2$.

30. Wishing to find the distance between two objects inaccessible from each other, I took a station 562 yards from one, and 320 yards from the other; the angle which the objects subtended at it was 128° 4′; find the distance. Ans. 799.89.

31. A person standing on the edge of a river takes the elevation of a tower on the other side, close to the water, and finds it 55°; receding 30 yards in a direct line from the tower, he finds the elevation to be 48°; find the breadth of the river. Ans. 104.93 yards.

32. An object 6 feet high placed on the top of a tower subtends an angle $= \tan.^{-1} .015$, at a place whose horizontal distance from the foot of the tower is 100 feet, determine the tower's height.

33. From the top of a mountain a miles high, the visible horizon appeared depressed $A°$; required the diameter $(2r)$ of the earth, and the distance (d) of the horizon.

$$d = a \cot. \frac{A}{2}; \; r = a \cot. \frac{A}{2} . \cot. A.$$

Ex. Let $a = 3$ miles, $A = 2° \, 13' \, 27''$.
Then $d = 154.54$; $2r = 7958$ miles.

34. A person at a known distance from two towers, observes that their apparent altitudes are the same; he then walks a given distance towards them, till the angle of elevation of one is double that of the other: find the heights of the towers.

35. In the arc AB of a circle centre O,

AC is taken $=$ sine AB; then will

sector COB $=$ segment ACB.

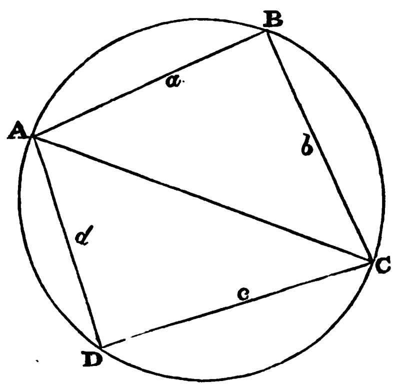
36. The shadows of two vertical walls, at right angles to each other, and which are a and b in height, are observed at 12 o'clock to be ma and mb in breadth: show, if α be the sun's altitude, and β the inclination of the first wall to the meridian,

$$\cot. \alpha = m\sqrt{2}; \; \cot. \beta = \frac{ma}{b}.$$

CHAPTER V.

AREA OF QUADRILATERAL INSCRIBED IN A CIRCLE. AREA AND PERIMETER OF CIRCUMSCRIBED AND INSCRIBED EQUILATERAL POLYGONS.

86. Find the area of the quadrilateral inscribed in a circle in terms of its sides.



Let a, b, c, d, be the sides of the quadrilateral $ABCD$, A, B, C, D, the four angles.

Then $ABCD = \Delta ABC + \Delta ADC$

$$= \frac{ab \,.\, \sin. B}{2} + \frac{cd \,.\, \sin. D}{2}.$$

But $B + D = \pi$; $\therefore \sin. D = \sin. (\pi - B) = \sin. B$;

$$\therefore \text{ area } = \frac{ab + cd}{2} \,.\, \sin. B.$$

To find $\sin. B$.

$$AC^2 = a^2 + b^2 - 2ab \,.\, \cos. B, \text{ also } = c^2 + d^2 - 2cd \,.\, \cos. D.$$

But $\cos. D = \cos. (\pi - B) = -\cos. B$;

$$\therefore\ c^2 + d^2 + 2cd \,.\, \cos. B = a^2 + b^2 - 2ab \,.\, \cos. B;$$

$$\therefore\ \cos. B = \frac{a^2 + b^2 - (c^2 + d^2)}{2(ab + cd)};$$

$$1 + \cos. B = \frac{(a^2 + b^2 + 2ab) - (c^2 - 2cd + d^2)}{2(ab + cd)} = \frac{(a + b)^2 - (c - d)^2}{2(ab + cd)};$$

$$\therefore\ (ab + cd) \,.\, (1 + \cos. B) = \frac{(a + b + c - d) \,.\, (a + b - c + d)}{2};$$

$$1 - \cos. B = \frac{c^2 + d^2 + 2cd - (a^2 - 2ab + b^2)}{2(ab + cd)} = \frac{(c + d)^2 - (a - b)^2}{2(ab + cd)};$$

$$\therefore\ (ab + cd) \,.\, (1 - \cos. B) = \frac{(c + d + a - b) \,.\, (c + d - a + b)}{2}.$$

Whence, if

$$2S = \text{the sum of the sides} = a + b + c + d;$$

$$\therefore (ab + cd) \,.\, (1 + \cos. B) = \frac{4(S - d) \,.\, (S - c)}{2} = 2(S - c) \,.\, (S - d);$$

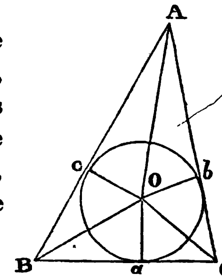
$$(ab + cd) \,.\, (1 - \cos. B) = \frac{4(S - b) \,.\, (S - a)}{2} = 2(S - a) \,.\, (S - b);$$

$$\therefore\ \overline{ab + cd}|^2 \,.\, \sin.^2 B = 4(S - a) \,.\, (S - b) \,.\, (S - c) \,.\, (S - d);$$

$$\therefore\ \text{area} = \frac{ab + cd}{2} \,.\, \sin. B = \sqrt{(S - a) \,.\, (S - b) \,.\, (S - c) \,.\, (S - d)}.$$

Cor. If $d = 0$, the quadrilateral becomes a triangle, and the area is $\sqrt{(S - a) \,.\, (S - b) \,.\, (S - c)\, S}$, which has been obtained before.

87. Find the area of a triangle in terms of the perimeter and the radius of the inscribed circle.



ABC the triangle, O the centre of the inscribed circle. Join Oa, Ob, Oc; a, b, c, being the points at which the sides of the triangle touch the circle. Hence Oa, Ob, Oc, are perpendicular to the sides.

Let (r) be the radius, and a, b, c, the sides of the triangle. Now

$$ABC = BOC + AOC + AOB = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} = \frac{r(a+b+c)}{2}$$

$$= S \,.\, r, \text{ where } S = \frac{a+b+c}{2};$$

or the area = product of the radius and semi-perimeter.

Prob. Given the angles; let the sides be required. Now, by Euclid, Book IV. Prop. IV.,

$$\angle\ OBa = \frac{B}{2}, \text{ and } \angle\ OCa = \frac{C}{2}.$$

$$\text{Now, } Ba = Oa \,.\, \text{cot.} \frac{B}{2};\ Ca = Oa \,.\, \text{cot.} \frac{C}{2};$$

$$\therefore\ a = Ba + aC = r\left\{\text{cot.} \frac{B}{2} + \text{cot.} \frac{C}{2}\right\}.$$

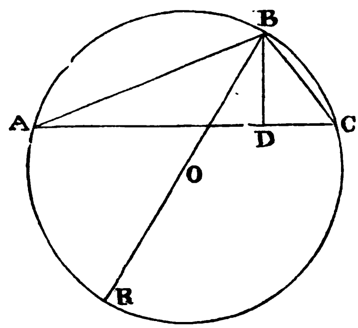
$$\text{Similarly, } b = Ab + bC = r\left\{\text{cot.} \frac{A}{2} + \text{cot.} \frac{C}{2}\right\}.$$

$$c = Ac + cB = r\left\{\text{cot.} \frac{A}{2} + \text{cot.} \frac{B}{2}\right\}.$$

Cor. $\because$ area $= \sqrt{S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)} = S\,.\,r$;

$$\therefore\ r = \sqrt{\frac{S-a)\,.\,(S-b)\,.\,(S-c)}{S}}.$$

88. Find the area of a triangle in terms of its sides and the radius (R) of the circumscribed circle.



ABC the triangle, BOR the diameter of the circle, BD perpendicular to AC.

Then Euclid, Book VI. Prop. C.

$AB \times BC = BR \,.\, BD$,

or $BD = \dfrac{c \,.\, a}{2R}$.

$$\text{But area} = \frac{AC \times BD}{2} = \frac{b \times BD}{2} = \frac{abc}{4R}.$$

Prob. To find the sides and area in terms of the angles and the radius.

$$AB = 2R \,.\, \text{sin.}\, \tfrac{1}{2} AOB = 2R \,.\, \text{sin.}\, C\,;$$

$$BC = 2R \,.\, \text{sin.}\, \tfrac{1}{2} BOC = 2R \,.\, \text{sin.} A\,;$$

$$\therefore \text{Area} = \frac{AB \,.\, BC \,.\, \text{sin.} B}{2} = 2R^2 . \,\text{sin.} A \,.\, \text{sin.} B \,.\, \text{sin.} C.$$

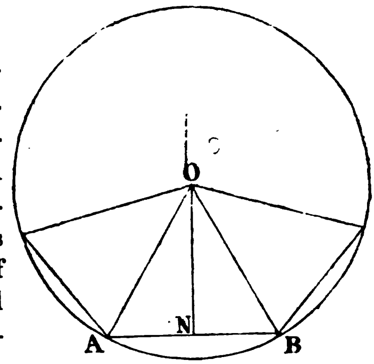
$$\text{Cor.}\ \because \frac{a+b+c}{2} r = \text{area} = \frac{abc}{4R}\,;$$

$$\therefore Rr = \frac{abc}{2(a+b+c)}.$$

89. Find the area of an equilateral polygon of (n) sides inscribed in a given circle, in terms of its radius.

AB one of the sides, O the centre of the circle.

Draw ON perpendicular to AB, and join OA, OB; then, by drawing lines from O to the angular points of the polygon, it may be divided into as many equal triangles as the figure has sides, and the area of the polygon will equal the sum of these triangles.



$$\therefore \text{ Polygon} = n \,.\, \Delta\, OAB = \frac{n\,AB\,.\,ON}{2} = n\,ON\,.\,AN,$$

The angles at the point $O = 4$ right angles $= 2\pi$;

$$\therefore\ AOB = \frac{2\pi}{n};\ AON = \frac{\pi}{n};\ \text{let } AO = r;$$

$$\therefore\ AN = r\,.\,\sin.\frac{\pi}{n};\ ON = r\,.\,\cos.\frac{\pi}{n};$$

$$\therefore\ \text{area} = nr^2\,.\,\sin.\frac{\pi}{n}\,.\,\cos.\frac{\pi}{n} = \frac{nr^2}{2}\,.\,\sin.\left(\frac{2\pi}{n}\right).$$

The perimeter $= n \times AB = 2n\,.\,AN = 2nr\,.\,\sin.\frac{\pi}{n}$.

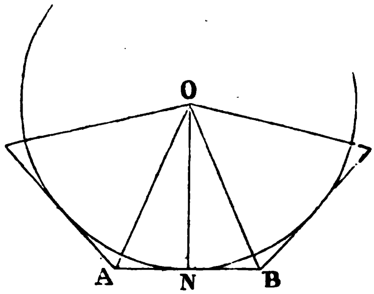
Ex. Find the area of a regular inscribed octagon.

$$\text{Here } n = 8,\ \frac{2\pi}{n} = \frac{360}{8} = 45.$$

$$\text{Area} = \frac{8r^2}{2}\,.\,\sin.45 = 4r^2\frac{1}{\sqrt{2}} = 2\sqrt{2}\,.\,r^2;$$

$$\therefore\ \text{area} : (\text{rad.})^2 :: 2\sqrt{2} : 1 :: \sqrt{8} : 1.$$

90. Find the area of a regular polygon of n sides described about a circle, in terms of its radius.



AB one of the sides, ON the radius, $\perp AB$.

$$\therefore \text{ polygon} = n \,.\, \Delta\, OAB = n \,.\, AN \,.\, ON.$$

$$\text{Let } ON = r; \ \angle AOB = \frac{2\pi}{n}; \ \therefore \angle AON = \frac{\pi}{n}.$$

$$\therefore AN = r \,.\, \tan. \frac{\pi}{n}; \ \therefore \text{ area} = nr^2 \,.\, \tan. \frac{\pi}{n}.$$

$$\text{The perimeter} = n \,.\, AB = 2nr \,.\, \tan. \frac{\pi}{n}.$$

91. To find the area of a regular polygon in terms of its side.

Let the given side $AB = a$, then (*see preceding figure*) bisect each of the angles A and B, by AO, BO, meeting in O; draw $ON \perp^r AB$.

$$\text{Then } \angle AOB = \frac{2\pi}{n}; \ \angle AON = \frac{\pi}{n}.$$

$$\text{Polygon} = n \,.\, \Delta\, OAB = n \,.\, AN \,.\, ON$$

$$= n \,.\, AN^2 \,.\, \frac{ON}{AN} = \frac{na^2}{4} \,.\, \cot. \frac{\pi}{n}.$$

Ex. (1.) Triangle $= \frac{3a^2}{4} \cdot \cot. \frac{\pi}{3} = \frac{a^2\sqrt{3}}{4}.$

(2.) Square $= \frac{4a^2}{4} \cdot \cot. \frac{\pi}{4} = a^2.$

(3.) Hexagon $= \frac{6a^2}{4} \cdot \cot. \frac{\pi}{6} = \frac{3a^2\sqrt{3}}{2}.$

92. To compare the areas of the inscribed and circumscribed polygon, and thence to find the area of the circle.

$$\frac{\text{Area of inscribed polygon}}{\text{Area of circumscribed polygon}} = \frac{\sin. \frac{\pi}{n} \cdot \cos. \frac{\pi}{n}}{\tan. \frac{\pi}{n}} = \frac{\cos.^2 \frac{\pi}{n}}{1}.$$

Now, as (n) increases, $\frac{\pi}{n}$ decreases; and when (n) becomes infinitely great, $\frac{\pi}{n}$ is infinitely small, and $\cos. \frac{\pi}{n} = 1$.

Therefore, when the number of sides is infinitely great, the inscribed and circumscribed polygons are equal. But the circle includes one of these areas, and is included by the other; when, therefore, the two limits become equal, it is also equal to either of them; or the area of a circle is equal to that of the inscribed polygon, whose sides are infinite in number.

Now, the area of inscribed polygon $= \frac{nr^2}{2} \cdot \sin. \frac{2\pi}{n}$ $= \pi r^2 \times \frac{\sin. \left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)}$; and when n is infinite, $\frac{2\pi}{n}$ and $\sin. \frac{2\pi}{n}$ are each $= 0$; we must therefore find the ratio between the sine and angle at the time they vanish.

But $A \;7\; \sin. A,\ \angle\ \tan. A.$

$$\text{But } \frac{\sin. A}{\tan. A} = \frac{\cos. A}{1} = \frac{1}{1}, \text{ if } A = 0,$$

or the sine and tangent are ultimately equal. Hence A, which lies between them, is ultimately equal to either of them.

$$\therefore \frac{\sin. \frac{2\pi}{n}}{\frac{2\pi}{n}} = 1 \text{ when } n = \text{infinity}.$$

Therefore the area of a polygon of an infinite number of sides inscribed in a circle $= \pi r^2$, which is therefore the area of the circle.

Hence, also, the perimeter of the circle $= 2nr . \sin. \frac{\pi}{n}$

$$= 2\pi r . \frac{\sin. \left(\frac{\pi}{n}\right)}{\left(\frac{\pi}{n}\right)} = 2\pi r, \text{ when } n \text{ is infinite.}$$

EXAMPLES.

(1.) If a, b, c, d, the sides of a quadrilateral inscribed in a circle, be in arithmetic progression, area $= \sqrt{a\,b\,c\,d}$.

(2.) Also, show that when a quadrilateral is capable of having one circle inscribed in and another described about it, area $= \sqrt{a\,b\,c\,d}$.

(3.) Given the perimeter of a right-angled triangle, and the radius of the inscribed circle; determine the sides.

(4.) Given the perimeter of the same triangle, and the radius of circumscribed circle; find the sides.

(5.) If (D) and (d) be the diameter of the two circles, and (a) (b) the sides which include the right angle; show that $D + d = a + b$.

(6.) The sides of a triangle are as 3, 5, 6; show that $R : r :: 15 : 4$.

(7.) If the side of a pentagon inscribed in a circle be 1, the radius $= \dfrac{\sqrt{5 + \sqrt{5}}}{\sqrt{10}}$.

(8.) Find the actual value of a side of a twenty-four-sided regular figure inscribed in a given circle.

(9.) If r be the radius of the inscribed circle,

$$\text{Area} = r^2 . \left\{\cot. \frac{A}{2} + \cot. \frac{B}{2} + \cot. \frac{C}{2}\right\}.$$

(10.) The perimeters of a triangle, a square, and a hexagon, each including the same area, are as $\sqrt[4]{27}$, $\sqrt[4]{16}$, $\sqrt[4]{12}$ respectively.

(11.) If a, b, c, be the sides, A, B, C, the angles of a triangle, and R be the radius of the circumscribing circle,

$$a \cos. A + b \cos. B + c \cos. C = 4 R \sin. A . \sin. B . \sin. C.$$

(12.) If O be the centre of the circle inscribed in the triangle ABC, $OA^2 + OB^2 + OC^2 = ab + ac + bc - 12 Rr$.

(13.) The area of a regular hexagon inscribed in a circle is a mean proportional between the areas of an inscribed and circumscribed equilateral triangle.

(14.) The square of the side of a pentagon inscribed in a circle is equal to the sum of the squares of the sides of a regular hexagon and decagon inscribed in the same circle.

(15.) If R and r be the radii of two circles, one described about, and the other inscribed in, a triangle, the distance between the centres $= \sqrt{R^2 - 2Rr}$.

(16.) Three equal circles touch each other; show that the space between them is nearly equal to the square described upon a fifth part of the diameter; find the area when the circles are unequal and the radii as the numbers 1, 2, 3.

(17.) Given $AB = a, BC = b, CD = c$, three sides of a quadrilateral, and $\angle ABC = \theta$, $\angle BCD = \phi$; show that twice the area $= ab \sin.\theta + cb \sin.\phi - ac \sin.(\theta + \phi.)$

(18.) $ABCD$ is the circumference of a circle, O its centre from C, draw a tangent to the circle meeting the radius OA produced in P, join PD; then if $CP = a$, $DP = b$, $\theta =$ the angle formed by DP and a tangent at D;

$r = \dfrac{a^2 - b^2}{2b} . \operatorname{cosec}. \theta$; r being the radius of the circle.

(19.) If $l =$ base of a polygon, a, b, c, &c. the sides a beginning from the base, α, β, γ, &c. the angles made by a, b, c, &c. with the base, then

$$l = a \cos. \alpha + b \cos. \beta + c \cos. \gamma + \&c.$$

(20.) If R and r be the radii of the circumscribed and inscribed circles,

$$r = 4R . \sin. \frac{A}{2} . \sin. \frac{B}{2} . \sin. \frac{C}{2} .$$

CHAPTER VI.

THEOREM OF DE MOIVRE. SERIES FOR SIN. ma, COS. ma, AND TAN. ma. SERIES FOR THE POWERS OF THE COSINE AND SINE IN TERMS OF THE COSINES AND SINES OF THE MULTIPLE ARC, &c.

93. THE important theorem of De Moivre is, that $(\cos. a + \sqrt{-1} \sin. a)^m = \cos. ma + \sqrt{-1} \sin. ma$, whether m be integral or fractional, positive or negative.

(1.) For $(\cos. a + \sqrt{-1} \sin. a)^2$

$$= \cos.^2 a - \sin.^2 a + 2\sqrt{-1} \sin. a . \cos. a$$
$$= \cos. 2a + \sqrt{-1} \sin. 2a.$$

And $\therefore$ $(\cos. a + \sqrt{-1} \sin. a)^3$

$$= (\cos. 2a + \sqrt{-1} \sin. 2a) \times (\cos. a + \sqrt{-1} \sin. a)$$
$$= \cos. 2a . \cos. a - \sin. 2a . \sin. a$$
$$+ \sqrt{-1} (\sin. 2a . \cos. a + \sin. a . \cos. 2a)$$
$$= \cos. 3a + \sqrt{-1} \sin. 3a.$$

Hence it would appear that

$$(\cos. a + \sqrt{-1} \sin. a)^m = \cos. ma + \sqrt{-1} \sin. ma.$$

But to prove this, let us assume that,

$$(\cos. a + \sqrt{-1} \sin. a)^{m-1} = \cos. \overline{m-1}\, a + \sqrt{-1} \sin. \overline{m-1}\, a.$$

$$\therefore (\cos. a + \sqrt{-1} \sin. a)^m = (\cos. \overline{m-1}\, a + \sqrt{-1} \sin. \overline{m-1}\, a)$$
$$\times (\cos. a + \sqrt{-1} \sin. a)$$
$$= \cos. \overline{m-1}\, a . \cos. a - \sin. \overline{m-1}\, a . \sin. a$$
$$+ \sqrt{-1} (\sin. \overline{m-1}\, a . \cos. a + \cos. \overline{m-1}\, a . \sin. a)$$
$$= \cos. ma + \sqrt{-1} \sin. ma.$$

Hence, if the assumption for the index $\overline{m-1}$ be correct, it holds for the index (m); *i. e.* for the next superior index. But it has been shown true when $m = 2$, and $m = 3$; it is therefore true when $m = 4$, $m = 5$, &c., and so, by successive inductions, may be shown to be universally true, so long as m is an integer.

(2.) Next let the index be fractional, and let $ma = nb$.

$$\therefore \cos. ma + \sqrt{-1} \sin. ma = \cos. nb + \sqrt{-1} \sin. nb,$$

$$\text{or } (\cos. a + \sqrt{-1} \sin. a)^m = (\cos. b + \sqrt{-1} \sin. b)^n.$$

$$\therefore (\cos. a + \sqrt{-1} \sin. a)^{\frac{m}{n}} = \cos. b + \sqrt{-1} \sin. b$$

$$= \cos. \frac{ma}{n} + \sqrt{-1} \sin. \frac{ma}{n}.$$

(3.) Let the index be negative $= -m$.

$$\text{Now } (\cos. a + \sqrt{-1} \sin. a)^{-m} = \frac{1}{(\cos. a + \sqrt{-1} \sin. a)^m}$$

$$= \frac{1}{\cos. ma + \sqrt{-1} \sin. ma} = \frac{\cos. ma - \sqrt{-1} \sin. ma}{\cos.^2 ma + \sin.^2 ma},$$

by multiplying the numerator and denominator by

$$\cos. ma - \sqrt{-1} \sin. ma;$$

$$\therefore (\cos. a + \sqrt{-1} \sin. a)^{-m} = \cos. ma - \sqrt{-1} \sin. ma$$

$$= \cos. (-ma) + \sqrt{-1} \sin. (-ma)$$

since $\cos. (-A) = \cos. A$, and $\sin. (-A) = -\sin. A$.

Therefore, whatever be the index (m),

$$(\cos. a + \sqrt{-1} \sin. a)^m = \cos. ma + \sqrt{-1} \sin. ma.$$

94. Expanding $(\cos. a + \sqrt{-1} . \sin. a)^m$ by the binomial theorem,

$$\cos. ma + \sqrt{-1} . \sin. ma = \cos.^m a$$

$$+ m \sqrt{-1} . \cos.^{m-1} a . \sin. a - m \frac{m-1}{2} . \cos.^{m-2} a . \sin.^2 a$$

$$- \frac{m \, \overline{m-1} \, \overline{m-2}}{1 . 2 . 3} \sqrt{-1} . \cos.^{m-3} a . \sin.^3 a$$

$$+ \frac{m \, \overline{m-1} \, \overline{m-2} \, \overline{m-3}}{1 . 2 . 3 . 4} . \cos.^{m-4} a . \sin.^4 a + \&c.$$

Hence, equating the real and imaginary parts of this equality respectively,

$$\sin. ma = m . \cos.^{m-1} a . \sin. a - \frac{m \, \overline{m-1} \, \overline{m-2}}{1 . 2 . 3} . \cos.^{m-3} a . \sin.^3 a + \&c.$$

$$\text{And } \cos. ma = \cos.^m a - m \frac{m-1}{2} . \cos.^{m-2} a . \sin.^2 a$$

$$+ \frac{m \, \overline{m-1} . \overline{m-2} \, \overline{m-3}}{1 . 2 . 3 . 4} . \cos.^{m-4} a . \sin.^4 a - \&c.$$

$$\therefore \frac{\sin. ma}{\cos. ma}, = \tan. ma$$

$$= \frac{m \sin. a . \cos. a^{m-1} a - \frac{m \, \overline{m-1} \, \overline{m-2}}{1 . 2 . 3} . \sin.^3 a . \cos.^{m-3} a + \&c.}{\cos.^m a - \frac{m \, \overline{m-1}}{1 . 2} . \cos.^{m-2} a . \sin.^2 a + \&c.}$$

$$= \frac{m \tan. a - \frac{m \, \overline{m-1} \, \overline{m-2}}{1 . 2 . 3} \tan.^3 a - \&c.}{1 - \frac{m \, \overline{m-1}}{1 . 2} \tan.^2 a + \&c.}$$

by dividing numerator and denominator by $\cos.^m a$.

By these formulas, for $\sin. ma$, and $\cos. ma$, we find the values of the sines and cosines of the multiple arc in terms of the powers of the sine and cosine of the simple arc; and the expression for the tangent ma is the tangent of the multiple arc in terms of the powers of the tangent of the simple arc.

95. Since

$$\text{Cos.}\, a - \sqrt{-1}\,.\,\sin. a = \frac{1}{\cos. a + \sqrt{-1}\,.\,\sin. a};$$

$$\therefore \text{ if } \cos. a + \sqrt{-1}\,.\,\sin. a = x,$$

$$\text{then } \cos. a - \sqrt{-1}\,.\,\sin. a = \frac{1}{x};$$

$$\therefore\ 2\cos. a = x + \frac{1}{x}, \text{ and } 2\sqrt{-1}\,.\,\sin. a = x - \frac{1}{x}.$$

$$x^m = (\cos. a + \sqrt{-1}\,.\,\sin. a)^m = \cos. ma + \sqrt{-1}\,.\,\sin. ma;$$

$$\frac{1}{x^m} = (\cos. a - \sqrt{-1}\,.\,\sin. a)^m = \cos. ma - \sqrt{-1}\,.\,\sin. ma.$$

$$\therefore\ x^m + \frac{1}{x^m} = 2\cos. ma,$$

$$\text{And } x^m - \frac{1}{x^m} = 2\sqrt{-1}\,.\,\sin. ma.$$

96. To find the powers of the cosines of the simple arc, in terms of the cosines of the multiple arc.

$$\text{Let } 2\cos. a = x + \frac{1}{x};$$

$$\therefore\ 2^n \cos.^n a = \left(x + \frac{1}{x}\right)^n = x^n + nx^{n-2} + n\,.\,\frac{n-1}{2}\,x^{n-4} + \text{\&c.}$$

$$+ n\,.\,\frac{n-1}{2}\,.\,\frac{1}{x^{n-4}} + n\,.\,\frac{1}{x^{n-2}} + \frac{1}{x^n}$$

$$=\left(x^n+\frac{1}{x^n}\right)+n\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+n\,.\,\frac{n-1}{2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)+\text{\&c.}$$

by collecting in pairs the terms which are equidistant from each end of the expansion,

$$=2\cos.na+2n\,.\,\cos.\overline{n-2}a+2n\,.\,\frac{\overline{n-1}}{2}\,.\,\cos.\overline{n-4}a+\text{\&c.}$$

If n be an even number, the number of terms in the expanded binomial is $n+1$, which is an odd number; and the middle term of the binomial, or $\left(\frac{n}{2}+1\right)^{\text{th}}$ term, which is the last term of the arranged expansion, will not contain x, for it is of the form $A\,.\,x^{\frac{n}{2}}\times\frac{1}{x^{\frac{n}{2}}}=A$.

To find A. In general the $(1+r)^{\text{th}}$ coefficient of the binomial $(a+b)^n=\dfrac{n\overline{n-1}\;\overline{n-2}\;\overline{n-3}\ldots\ldots\overline{n-r+1}}{1\,.\,2\,.\,3\,.\,4\;\ldots\ldots\;r}$;

$$\therefore\left(\frac{n}{2}+1\right)^{\text{th}}=\frac{n\,.\,\overline{n-1}\;\overline{n-2}\;\overline{n-3}\ldots\ldots\left(n-\frac{n}{2}+1\right)}{1\,.\,2\,.\,3\,.\,4\;\ldots\ldots\;\frac{n}{2}}$$

$$=\frac{n\,.\,\overline{n-1}\;\overline{n-2}\ldots\ldots\left(\frac{n}{2}+1\right)}{1\,.\,2\,.\,3\;\ldots\ldots\;\frac{n}{2}}.$$

and first dividing each side of the equation by 2,

$$2^{n-1}\,.\,\cos.^n a=\cos.na+n\,.\,\cos.\overline{n-2}a+n\,.\,\frac{\overline{n-1}}{2}\,.\,\cos.\overline{n-4}a+\text{\&c.}$$

$$+\frac{n\,.\,\overline{n-1}\;\overline{n-2}\ldots\ldots\overline{\frac{n}{2}+1}}{1\,.\,2\,.\,3\;\ldots\ldots\frac{n}{2}}\times\frac{1}{2}.$$

Next, let (n) be odd; therefore $\overline{n+1}$, the number of terms, is even; and there will be $\frac{n+1}{2}$ pairs of terms in the expansion; and therefore $\frac{n+1}{2}$ terms in the series containing the cosines of the multiple angles, and the last

$$\text{term will} = \frac{n\,\overline{n-1}\,\overline{n-2}\ldots\left(n-\frac{n-1}{2}+1\right)}{1\,.\,2\,.\,3\ \ldots\ \frac{n-1}{2}}\left(x+\frac{1}{x}\right).$$

$$= \frac{n\,\overline{n-1}\,\overline{n-2}\ldots\frac{n+3}{2}}{1\,.\,2\,.\,3\ \ldots\frac{n-1}{2}}\,.\,2\cos. a.^*$$

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* The middle term of the binomial, when n is even, may be thus expressed.

$$\frac{n\,\overline{n-1}\,\overline{n-2}\ldots\overline{\tfrac{n}{2}+1}}{1\,.\,2\,.\,3\ \ldots\tfrac{n}{2}} = 2^{\frac{n}{2}}\left\{\frac{n\,\overline{n-1}\,\overline{n-2}\ldots\overline{\tfrac{n}{2}+1}}{2\,.\,4\,.\,6\ \ldots n}\right\}$$

$$= 2^{\frac{n}{2}}\left\{\frac{n\,\overline{n-1}\,\overline{n-2}\ldots\overline{\tfrac{n}{2}+1}}{1\,.\,2\,.\,3\ \ldots\tfrac{n}{2}}\right\}\times\frac{1\,.\,3\,.\,5\ \ldots\ \overline{n-1}}{\left(\tfrac{n}{2}+1\right)\ \ldots\ \overline{n-1}\,n}$$

$$= 2^{\frac{n}{2}}\left\{\frac{1\,.\,3\,.\,5\ \ldots n-1}{1\,.\,2\,.\,3\ \ldots\tfrac{n}{2}}\right\}.$$

Next, when n is odd. The coefficient of one of the middle terms

$$= \frac{n\,\overline{n-1}\,\overline{n-2}\ldots\frac{n+3}{2}}{1\,.\,2\,.\,3\ \ldots\frac{\overline{n-1}}{2}} = 2^{\frac{n-1}{2}}\left\{\frac{n\,\overline{n-1}\,\overline{n-2}\ldots\frac{n+3}{2}}{2\,.\,4\,.\,6\ \ldots\overline{n-1}}\right\}$$

$$= 2^{\frac{n-1}{2}}\left\{\frac{n\,\overline{n-1}\,\overline{n-2}\ldots\frac{n+3}{2}\times 1\,.\,3\,.\,5\ldots n}{1\,.\,2\,.\,3\ \ldots\frac{n+1}{2}\times\frac{n+3}{2}\ldots\ldots\overline{n-1}\,n}\right\}$$

$$= 2^{\frac{n-1}{2}}\left\{\frac{1\,.\,3\,.\,5\ \ldots n}{1\,.\,2\,.\,3\ \ldots\frac{n+1}{2}}\right\}.$$

And again, dividing by (2), we have, 2^{n-1} $(\cos.^n a)$

$$=\cos. na + n.\cos.\overline{n-2}a + n\frac{n-1}{2}.\cos.\overline{n-4}a + \&c.$$

$$+\frac{n\,\overline{n-1}\,\overline{n-2}\ldots.\frac{n+3}{2}}{1\,.\,2\,.\,3\quad\ldots.\frac{n-1}{2}}.\cos. a.$$

Hence, $2\cos.^2 a = \cos. 2a + 2 \times \frac{1}{2} = \cos. 2a + 1.$

$$2^2\cos.^3 a = \cos. 3a + 3\cos. a\,.$$

$$2^3\cos.^4 a = \cos. 4a + 4\cos. 2a + \frac{4\,.\,3}{1\,.\,2}\times\tfrac{1}{2}$$

$$= \cos. 4a + 4\cos. 2a + 3.$$

$$2^4\cos.^5 a = \cos. 5a + 5\cos. 3a + 10\cos. a.$$

97. To find the powers of the sine of the simple angle, in terms of the sines or cosine of the multiple angle.

$$2\sqrt{-1}\,.\,\sin. a = x - \frac{1}{x};$$

$$\therefore\ 2^n(\sqrt{-1})^n\,.\,(\sin. a)^n = \left(x - \frac{1}{x}\right)^n.$$

The values of the powers of $\sqrt{-1}$ recur after the fourth.

For $\overline{\sqrt{-1}}|^2 = -1.$

$$\overline{\sqrt{-1}}|^3 = \overline{\sqrt{-1}}|^2 \times \sqrt{-1} = -\sqrt{-1}.$$

$$\overline{\sqrt{-1}}|^4 = (\sqrt{-1})^2 \times (\sqrt{-1})^2 = -1 \times -1 = 1.$$

$$\overline{\sqrt{-1}}|^5 = \overline{\sqrt{-1}}|^4 \times \sqrt{-1} = \sqrt{-1}.$$

the last value reproduces $\sqrt{-1}$; also $(\sqrt{-1})^{4m} = \overline{(\sqrt{-1})^4}|^m = 1^m = 1.$

Hence there are four cases,

(n) may be even, and of one of the forms, $4m$ or $4m+2$;

(n) may be odd, and of one of the forms, $4m+1$ or $4m+3$.

First, let (n) be odd, and therefore the number of terms even; also the last term of $\left(x-\frac{1}{x}\right)^n$ will be negative, since the terms are alternately positive and negative.

Then $2^n(\sqrt{-1})^n \sin.^n a$

$$=\left(x^n-\frac{1}{x^n}\right)-n\left(x^{n-2}-\frac{1}{x^{n-2}}\right)+n\frac{\overline{n-1}}{2}\left(x^{n-4}-\frac{1}{x^{n-4}}\right)-\&c.$$

$$=2\sqrt{-1}\,\{\sin. na-n\,.\,\sin.\overline{n-2}\,a+\&c.\}$$

Let n be of the form $\overline{4m+1}$;

$$\therefore\ (\sqrt{-1})^{4m+1}=\overline{\sqrt{-1}}|^{4m}\times\sqrt{-1}=\sqrt{-1}.$$

Let n be of the form $\overline{4m+3}$;

$$\therefore\ \overline{\sqrt{-1}}|^{4m+3}=(\sqrt{-1})^{4m}\times\overline{\sqrt{-1}}|^{3}=-\sqrt{-1}.$$

substituting, and dividing by $2\sqrt{-1}$,

$$\pm 2^{n-1}\sin.^n a=\sin. na-n\,.\,\sin.\overline{n-2}\,a+n\,.\,\frac{\overline{n-1}}{2}\,.\,\sin.\overline{n-4}\,a-\&c.$$

Next, let n be even, and therefore $\overline{n+1}$, the number of terms, odd; then the last term of $\overline{x-\frac{1}{x}}|^n$ will be positive, and $2^n\,\overline{\sqrt{-1}}|^n\,.\,\sin.^n a$.

$$=\left(x^n+\frac{1}{x^n}\right)-n\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+n\frac{\overline{n-1}}{2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)+\&c.$$

$$=2\left\{\cos. na-n\cos.\overline{n-2}\,a+n\frac{\overline{n-1}}{2}\,.\,\cos.\overline{n-4}\,a-\&c.\right\}$$

Now, if n be of the form $4m$, $(\sqrt{-1})^{4m}=1$.

If n be of the form $4m+2$, $(\sqrt{-1})^{4m+2} = \overline{\sqrt{-1}}|^{4m}$
$$\times (\sqrt{-1})^2 = -1.$$

Hence, substituting and dividing by 2,

$$\pm 2^{n-1} \sin.^n a = \cos. na - n \cos. \overline{n-2}a + n\frac{\overline{n-1}}{2} \cos. \overline{n-4}a - \&c.$$

The last terms may be determined in the same manner as the last term of $\cos.^n a$.

Hence, $-2\sin.^2 a = \cos. 2a - 1$;
$$\therefore\ 2\sin.^2 a = 1 - \cos. 2a.$$
$$-2^2 \sin.^3 a = \sin. 3a - 3\sin. a;$$
$$\therefore\ 2^2 \sin.^3 a = 3\sin. a - \sin. 3a.$$
$$2^3 \sin.^4 a = \cos. 4a - 4\cos. 2a + 3.$$
$$2^4 \sin.^5 a = \sin. 5a - 5\sin. 3a + 10 \sin. a.$$

98. Find the tangent of the sum of any angles in terms of the tangents of the simple angles.

By multiplication,
$$(\cos. a + \sqrt{-1}\sin. a) \times (\cos. b + \sqrt{-1}\sin. b)$$
$$= \cos. a . \cos. b - \sin. a . \sin. b + \sqrt{-1}(\sin. a . \cos. b + \sin. b . \cos. a)$$
$$= \cos. (a+b) + \sqrt{-1}\sin. (a+b).$$

For (b) put $(b+c)$.

$\therefore\ \cos. (a+b+c) + \sqrt{-1}\sin. (a+b+c)$
$= (\cos. a + \sqrt{-1}\sin. a) \times (\cos. \overline{b+c} + \sqrt{-1}\sin. \overline{b+c})$
$= (\cos. a + \sqrt{-1}\sin. a) \times (\cos. b + \sqrt{-1}\sin. b)$
$\times\ (\cos. c + \sqrt{-1}\sin. c)$;

Write $c+d$ for c, $\overline{d+e}$ for d, and so on,

$$\therefore \cos.(a+b+c+d+\&c. \text{ to } n \text{ terms})$$
$$+\sqrt{-1}\sin.(a+b+c+\&c. \text{ to } n \text{ terms})$$
$$=(\cos.a+\sqrt{-1}\sin.a)\times(\cos.b+\sqrt{-1}\sin.b)$$
$$\times(\cos.c+\sqrt{-1}\sin.c)\ldots\text{ to } n \text{ factors,}$$
$$=\cos.a\,.\,\cos.b\,.\,\cos.c\ldots(1+\sqrt{-1}\tan.a)$$
$$(1+\sqrt{-1}\tan.b).(1+\sqrt{-1}\tan.c)\ \&c.$$

Let T_1 = sum of the tangents.
T_2 = sum of the product of two and two.
T_3 = sum of the product of three and three.
T_4 = sum of the product of four and four.
&c. &c.

$$\therefore (\cos.a+b+c+d+\&c.)+\sqrt{-1}\,.\,\sin.(a+b+c+d+\&c.)$$
$$=\cos.a\,.\,\cos.b\,.\,\cos.c\,.\,\&c.$$
$$\{1+\sqrt{-1}\,.\,T_1-T_2-\sqrt{-1}\,.\,T_3+T_4+\sqrt{-1}\,.\,T_5-\&c.\}$$

by actual multiplication;

$$=\cos.a\,.\,\cos.b\,.\,\cos.c\,.\,\&c.\ \{1-T_2+T_4-\&c.\}$$
$$+\sqrt{-1}\,.\,\cos.a\,.\,\cos.b\,.\,\cos.c\,.\,\&c.\ \{T_1-T_3+T_5-\&c.$$

Hence, equating the impossible parts, and then the possible parts,

$$\sin.(a+b+c+d+\&c.)$$
$$=\cos.a\,.\,\cos.b\,.\,\cos.c\,.\,\&c.\ \{T_1-T_3+T_5-\&c.\}\ (1).$$
$$\text{and } \cos.(a+b+c+d+\&c.)$$
$$=\cos.a\,.\,\cos.b\,.\,\cos.c\,.\,\&c.\ \{1-T_2+T_4-\&c.\}\ (2).$$

Therefore, by division,

$\tan.(a+b+c+d+\&c.)=\dfrac{T_1-T_3+T_5-\&c.}{1-T_2+T_4-\&c.}$, the expression required.

Cor. From the equations (1) and (2) we also have the values of the sine and cosine of the sum of any number of angles.

Thus, to find $\sin.(a+b+c)$,

$$T_1 = \tan. a + \tan. b + \tan. c = \frac{\sin. a}{\cos. a} + \frac{\sin. b}{\cos. b} + \frac{\sin. c}{\cos. c}.$$

$$T_3 = \tan. a \,.\, \tan. b \,.\, \tan. c = \frac{\sin. a}{\cos. a} \times \frac{\sin. b}{\cos. b} \times \frac{\sin. c}{\cos. c}.$$

$$\therefore\ \sin.(a+b+c) = \cos. b \,.\, \cos. c \,.\, \sin. a + \cos. a \,.\, \cos. c \,.\, \sin. b$$
$$+ \cos. a \,.\, \cos. b \,.\, \sin. c + \sin. a \,.\, \sin. b \,.\, \sin. c,$$

$$\text{and } \cos.(a+b+c) = \cos. a \,.\, \cos. b \,.\, \cos. c - \cos. a \,.\, \sin. b \,.\, \sin. c$$
$$- \cos. b \,.\, \sin. a \,.\, \sin. c - \cos. c \,.\, \sin. a \,.\, \sin. b.$$

99. By Art. 94,

$$\cos. ma = \cos.^m a \left\{1 - m\frac{\overline{m-1}}{2} \tan.^2 a + \frac{m\overline{m-1}\,\overline{m-2}\,\overline{m-3}}{1\,.\,2\,.\,3\,.\,4} \tan.^4 a - \&c.\right\}$$

$$\sin. ma = \sin.^m a \left\{m \tan. a - \frac{m\overline{m-1}\,\overline{m-2}}{1\,.\,2\,.\,3} \tan.^3 a + \&c.\right\}$$

Let $ma = x$; $\therefore\ m = \frac{x}{a}$.

Therefore,

$$\cos. x = \cos.^m a \left\{1 - \frac{x\overline{x-a}}{1\,.\,2} \,.\, \frac{\tan.^2 a}{a^2} + \frac{x\overline{x-a}\,.\,\overline{x-2a}\,.\,\overline{x-3a}}{1\,.\,2\,.\,3\,.\,4} \,.\, \frac{\tan.^4 a}{a^4} - \&c.\right\}$$

$$\sin. x = \cos.^m a \left\{x \,.\, \frac{\tan. a}{a} - \frac{x\overline{x-a}\,.\,\overline{x-2a}}{1\,.\,2\,.\,3} \,.\, \frac{\tan.^3 a}{a^3} + \&c.\right\}$$

Now, if x be finite, a may have any value whatever, so long as the equation, $m = \frac{x}{a}$ can be satisfied; and if $a = 0$, then, if m be infinite, x is still finite; let, therefore $a = 0$.

Then

$$\frac{\tan. a}{a} = 1\,;\ \cos. a = \cos. 0 = 1\,;\ \text{and} \ \therefore\ \cos.^{m} a = 1,$$

$$\text{whence } \cos. x = 1 - \frac{x^2}{1\,.\,2} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} - \&c.$$

$$\sin. x = x - \frac{x^3}{2\,.\,3} + \frac{x^5}{2\,.\,3\,.\,4\,.\,5} - \&c.$$

two remarkable and important theorems, by which the sine and cosine are expressed in terms of the angle.

It was shown in Art. 55, that the error by making $\sin. 1' = 1'$ was less than the cube of the minute; now from the theorem for $\sin. x$, we find the error by excess to be $= \dfrac{x^3}{2\,.\,3}\left\{1 - \dfrac{x^2}{4\,.\,5} + \&c.\right\}$ or less than $\frac{1}{6}$th part of the cube of the number which expresses the minute.

100. If x be a very small quantity, we may obtain the following useful theorem.

$$\text{Sin.}\, x = x - \frac{x^3}{2\,.\,3}, \text{ very nearly}$$

$$= x\left(1 - \frac{x^2}{2\,.\,3}\right) = x\left(1 - \frac{x^2}{2}\right)^{\frac{1}{3}}$$

$$= x\,(\cos. x)^{\frac{1}{3}}.$$

$$\therefore\ \log. \sin. x = \log. x + \frac{1}{3}\log. \cos. x.$$

101. Since by algebra we know that when

$$e = 2.71828\ldots.$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\,.\,3} + \frac{x^4}{2\,.\,3\,.\,4} + \&c.;$$

$\therefore$ writing $x\sqrt{-1}$ and $-x\sqrt{-1}$, successively for x;

$$\therefore\ e^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{2} - \frac{x^3\sqrt{-1}}{2.3} + \frac{x^4}{2.3.4} + \&c.$$

$$e^{-x\sqrt{-1}} = 1 - x\sqrt{-1} - \frac{x^2}{2} + \frac{x^3\sqrt{-1}}{2.3} + \frac{x^4}{2.3.4} - \&c.;$$

$$\therefore\ e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2\left\{1 - \frac{x^2}{1.2} + \&c.\right\} = 2\cos. x$$

$$e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2\sqrt{-1}\left\{x - \frac{x^3}{2.3} + \&c.\right\} = 2\sqrt{-1}.\sin. x;$$

$$\therefore\ \cos. x \pm \sqrt{-1}\,.\,\sin. x = e^{\pm x\sqrt{-1}}.$$

Cor. Hence $\sqrt{-1}\,.\,\tan. x = \dfrac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1}$.

102. Put nx for x, n being any number whatever.

$$\text{Cos.}\, nx \pm \sqrt{-1}\,.\,\sin. nx = e^{\pm nx\sqrt{-1}} = (e^{\pm x\sqrt{-1}})^n$$
$$= (\cos. x \pm \sqrt{-1}\,.\,\sin. x)^n;$$

which is another proof of De Moivre's formula.

103. Find the cosine of the multiple angle in terms of the powers of the cosine of the simple angle.

If $2\cos. a = x + \frac{1}{x}$, $2\cos. na = x^n + \frac{1}{x^n}$, our object is to find $\cos. na$ in terms of the powers of $\cos. a$, or of the powers of $\left(x + \frac{1}{x}\right)$.

$$\text{Now } \left(x + \frac{1}{x}\right)^n = x^n + \frac{1}{x^n} + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \&c.;$$

$$\therefore\ x^n + \frac{1}{x^n} = \overline{x + \frac{1}{x}}\Big|^n - n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) - \&c.$$

Similarly,

$$x^{n-2} + \frac{1}{x^{n-2}} = \overline{x + \frac{1}{x}}\Big|^{n-2} - \overline{n-2}\left\{x^{n-4} + \frac{1}{x^{n-4}}\right\} - \&c.$$

Hence, by substitution, we find that we may put

$$x^n + \frac{1}{x^n} = \overline{x + \frac{1}{x}}\Big|^{n} + A\left(x+\frac{1}{x}\right)^{n-2} + B\left(x+\frac{1}{x}\right)^{n-4} + C\left(x+\frac{1}{x}\right)^{n-6} + \&c.$$

$$= x^n + nx^{n-2} + n\frac{\overline{n-1}}{2}x^{n-4} + \frac{n\,\overline{n-1}\,\overline{n-2}}{1\,.\,2\,.\,3}x^{n-6} + \&c.$$

$$+ A\left\{x^{n-2} + \overline{n-2}\,x^{n-4} + \frac{\overline{n-2}\,\overline{n-3}}{1\,.\,2}x^{n-6} + \&c.\right\}$$

$$+ B\{x^{n-4} + \overline{n-4}\,x^{n-6} + \&c.\}$$

$$+ C\{x^{n-6} + \&c.\}$$

$$= x^n + (A+n)x^{n-2} + \left(n\frac{\overline{n-1}}{2} + A\overline{n-2} + B\right)x^{n-4}$$

$$+ \left(\frac{n\,\overline{n-1}\,\overline{n-2}}{1\,.\,2\,.\,3} + \frac{A\,.\,\overline{n-2}\,\overline{n-3}}{1\,.\,2} + \overline{n-4}\,.\,B + C\right)x^{n-6} + \&c.$$

$$\therefore A + n = 0;\ \therefore A = -n.$$

$$n\frac{\overline{n-1}}{2} + A\,.\,\overline{n-2} + B = 0;\ \therefore B = n\left\{n-2-\frac{n-1}{2}\right\} = n\frac{\overline{n-3}}{2}.$$

$$C + B\,.\,\overline{n-4} + \frac{A\,.\,\overline{n-2}\,\overline{n-3}}{2} + \frac{n\,\overline{n-1}\,\overline{n-2}}{1\,.\,2\,.\,3} = 0.$$

$$\therefore C = -\frac{n}{2}\left\{\overline{n-4}\,\overline{n-3} - \overline{n-2}\,\overline{n-3} + \frac{\overline{n-1}\,\overline{n-2}}{3}\right\}$$

$$= -\frac{n}{2}\,.\,\frac{n^2-9n+20}{3} = -\frac{n\,(n-4)\,(n-5)}{2\,.\,3}.$$

$$\therefore \cos. na = \tfrac{1}{2}\left\{(2\cos.a)^n - n(2\cos.a)^{n-2} + n\frac{n-3}{2}(2\cos.a)^{n-4} - \frac{n\,\overline{n-4}\,\overline{n-5}}{1\,.\,2\,.\,3}(2\cos.a)^{n-6} - \&c.\right\}$$

whence
$$\begin{aligned}\cos. 2a &= 2\cos.^2 a - 1;\\ \cos. 3a &= 4\cos.^3 a - 3\cos. a;\\ \cos. 4a &= 8\cos.^4 a - 8\cos.^2 a + 1;\\ \cos. 5a &= 16\cos.^5 a - 20\cos.^3 a + 10\cos. a;\\ \&c. &\quad \&c.\end{aligned}$$

COR. 1. The general term of the series for $\cos. na$ is

$$\pm \tfrac{1}{2}\left(\frac{n\,(n-m-1)\,(n-m-2)\ldots(n-2m+1)}{1\,.\quad 2\quad .\quad 3\quad \ldots\quad m}\right)(2\cos.a)^{n-2m}.$$

(1.) Let (n) be even, then the last term, where $n = 2m$, and therefore $n - m = m$, is

$$\pm \tfrac{1}{2}\,\frac{2m\;\overline{m-1}\;\overline{m-2}\ldots\ldots 2\,.\,1}{1\,.\,2\,.\,3\ldots\ldots\overline{m-1}\,m} = \pm \tfrac{1}{2}\,2 = \pm 1.$$

(2.) Let (n) be odd, then the last term, where $n = 2m + 1$, and therefore $n - m - 1 = m$, is

$$\pm \tfrac{1}{2}\,\frac{n\;m\;\overline{m-1}\;\overline{m-2}\ldots\ldots 2}{1.2\,.\,3\,.\,4\ldots\ldots m}(2\cos.a) = \pm n\cos.a.$$

If, when (n) is even, we put $n = 2m + 2$,
(n) is odd, ... $n = 2m + 3$,

the last terms but one are respectively

$$\pm \frac{n^2}{1.2}\cos.^2 a, \text{ and } \pm \frac{n\,(n^2-1)}{1.2\,.\,3}\cos.^3 a.$$

Therefore, when (n) is even,

$$\cos. na = \tfrac{1}{2}\left\{(2\cos.a)^n - n(2\cos.a)^{n-2} + \&c.\right\} \pm \frac{n^2}{1.2}\cos.^2 a \mp 1.$$

And, when (n) is odd,

$$\cos. n a = \tfrac{1}{2}\{\overline{2\cos. a}|^{n} - n(2\cos. a)^{n-2} + \&c.\} \pm \frac{n\, \overline{n^2-1}}{1.2\,.\,3}\cos.^3 a$$

$$\pm\, n \cos. a.$$

Cor. 2. These series may be written under another form, by reversing the order of the terms, the coefficients being computed by the general expression; thus, if n be even,

$$\cos. n a = \pm \left(1 - \frac{n^2}{1.2}\cos.^2 a + \frac{n^2\, \overline{n^2-4}}{1.2.3.4}\cos.^4 a - \&c.\right)$$

and n odd,

$$\cos. n a = \pm\left(n\cos. a - \frac{n(n^2-1)\cos.^3 a}{1.2\,.\,3} + \frac{n(n^2-1)\,(n^2-9)\cos.^5 a}{1.2\,.\,3\,.\,4\,.\,5} - \&c.\right)$$

The upper signs are to be used when n is of either of the forms $4m$ or $4m+1$, and the lower when of either of the forms $4m+2$ or $\overline{4m+3}$.

104. To sum the series $\sin. a + \sin.(a+b) + \sin.(a+2b) + \&c. + \sin.(a+\overline{n-1}\,b)$.

Since $\cos. B - \cos. A = 2\sin. \frac{A+B}{2}\sin. \frac{A-B}{2}$;

$$\therefore\ \cos.\left(a-\frac{b}{2}\right) - \cos.\left(a+\frac{b}{2}\right) = 2\sin. a\,.\,\sin. \frac{b}{2};$$

$$\cos.\left(a+\frac{b}{2}\right) - \cos.\left(a+\frac{3b}{2}\right) = 2\sin.(a+b)\sin. \frac{b}{2};$$

$$\cos.\left(a+\frac{3b}{2}\right) - \cos.\left(a+\frac{5b}{2}\right) = 2\sin.(a+2b)\sin. \frac{b}{2};$$

&c. &c. &c.

$$\cos.\left(a+\frac{2n-3}{2}b\right) - \cos.\left(a+\frac{2n-1}{2}b\right) = 2\sin.(a+\overline{n-1}\,b)\sin. \frac{b}{2}.$$

Whence, by addition, and making

$$M = \sin. a + \sin. (a + b) + \sin. (a + 2b) + \&c.$$

$$\cos. \left(a - \frac{b}{2}\right) - \cos. \left(a + \frac{2n-1}{2}\right) b = 2M . \sin. \frac{b}{2};$$

$$\text{or } 2 \sin. \left(a + \frac{n-1}{2} b\right) \sin. \frac{nb}{2} = 2M . \sin. \frac{b}{2};$$

$$\therefore M = \frac{\sin. \left(a + \frac{n-1}{2} b\right) \sin. \left(\frac{nb}{2}\right)}{\sin. \frac{b}{2}}$$

COR. If $b = 2a$; $a + \frac{n-1}{2} b = na$; $\frac{nb}{2} = na$.

$$\therefore \sin. a + \sin. 3a + \sin. 5a + \&c. \sin. \overline{2n-1}\, a = \frac{\sin.^2 na}{\sin. a}.$$

105. Similarly may be found the sum of

$$\cos. a + \cos. (a + b) + \cos. (a + 2b) + \&c. + \cos. (a + \overline{n-1}) b.$$

For $\sin. \left(a + \frac{b}{2}\right) - \sin. \left(a - \frac{b}{2}\right) = 2 \sin. \frac{b}{2} . \cos. a$; and substituting $a + b$, $a + 2b$, &c. for a, the sum

$$= \frac{\cos. \left(a + \frac{(n-1)b}{2}\right) . \sin. \frac{nb}{2}}{\sin. \frac{b}{2}}.$$

106. The following series may prove useful exercises.

(1.) $\text{Tan}. a + 2 \tan. 2a + 2^2 \tan. 2^2 a$ to n terms

$$= \cot. a - 2^n . \cot. 2^n a.$$

(2.) $\frac{1}{2} \tan. \frac{a}{2} + \frac{1}{2^2} \tan. \frac{a}{2^2} + \frac{1}{2^3} \tan. \frac{a}{2^n} + \&c.$

$$= \frac{1}{2^n} \cot. \frac{a}{2^n} - \cot. a.$$

(3.) Cosec. a + cosec. $2a$ + cosec. $2^2 a$ + &c. to n terms,

$$= \cot. \frac{a}{2} - \cot. 2^{n-1} a.$$

In (1) we see that $\tan. a = \cot. a - 2 \cot. 2a$.

In (2), $\frac{1}{2} \tan. \frac{a}{2} = \frac{1}{2} \cot. \frac{a}{2} - \cot. a$.

In (3), $\text{cosec.}\, a = \cot. \frac{a}{2} - \cot. a$.

(4.) $\text{Cosec.} \frac{a}{2} + \text{cosec.} \frac{a}{2^2} + \text{cosec.} \frac{a}{2^3} + \text{\&c.}$

$$= \cot. \frac{a}{2^{n+1}} - \cot. \frac{a}{2}.$$

(5.) $2^n \sin. \frac{a}{2^n} = \sin. a \,.\, \sec. \frac{a}{2} \,.\, \sec. \frac{a}{2^2} \ldots \sec. \frac{a}{2^n}$.

(6.) Hence $a = \sin. a \,.\, \sec. \frac{a}{2} \,.\, \sec. \frac{a}{2^2} \ldots$ if n be infinite.

(7.) $1 + x \cos. a + x^2 \cos. 2a + x^3 \cos. 3a + \text{\&c.}$

$$+ x^{n-1} . \cos. (n-1) a$$

$$= \frac{x^{n+1} . \cos. \overline{n-1}\, a - x^n . \cos. na - x \cos. a + 1}{x^2 - 2x \cos. a + 1}.$$

(8.) If $y = a \sin. x - \frac{a^2}{2} \sin. 2x + \frac{a^3}{3} \sin. 3x - \text{\&c. in inf}^{\text{m.}}$

$$\tan. y = \frac{a \sin. x}{1 + a \cos. x}.$$

SPHERICAL

TRIGONOMETRY.

I

SPHERICAL TRIGONOMETRY.

CHAPTER I.

1. A SPHERE is a solid bounded by a curve surface, every point of which is equidistant from a point within it, called the centre.

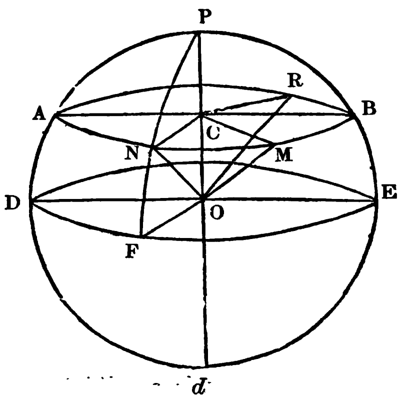
2. The radius is a straight line drawn from the centre to any point in the circumference. Any straight line drawn from a point in the surface, through the centre, to meet the surface again, is called a diameter.

All the radii of a sphere are equal, and the diameter is double of the radius.

3. The sphere may be conceived to be generated by the revolution of a semicircle round its diameter; and as all lines perpendicular to this diameter will necessarily describe circles, every section of the sphere made by a plane which is perpendicular to the diameter will be a circle; and since we may assume the generating circle to be in any part of the sphere, we may conclude that every section of a sphere made by a plane is a circle.

But the following is a direct proof of this proposition.

4. Every section of a sphere made by a plane is a circle.



Let AMB be the section made by the plane. O the centre of the sphere.

Draw OC perpendicular to the plane AMB, meeting it in C; and from C draw any straight lines, CM, CN, CR, to points M, N, R, in the section AMB. Then, since in the triangles OCM, OCN, OCR; OC is common, and each of the angles OCM, OCN, OCR, is a right angle; also the hypothenuses, OM, ON, OR, being radii of the sphere, are equal; therefore CN, CM, and CR, are equal, and $\therefore$ (Euclid, Book III. Prop. 9.) AMB is a circle, of which the centre is C.

5. If the cutting plane passed through O, the radius of the circle would be the radius of the sphere; such a circle is called a *great circle*, and it is obvious that all great circles are equal.

If the plane does not pass through the centre of the sphere, the section is called a small circle.

6. Hence, since great circles are sections made by planes passing through the centre of the sphere, the diameter will be the common intersection of two great circles; these, therefore, bisect each other.

7. An arc of a great circle may always be drawn through two given points upon the surface of a sphere. For a plane can be made to pass through three points, and the centre of the sphere and the two given points furnish the requisite conditions for a plane to be drawn.

8. The points P and p, at which OC, produced both ways, meets the surface of the sphere, are called poles of the circle ANB. They are equidistant from every point in the circumference of ANB.

For, (first considering the point P,) since AC and CN are equal, and CP is common, and $\angle PCA = \angle PCN$, each being a right angle, therefore the chord $AP =$ chord NP, and therefore arc $PA =$ arc PN.

And similarly $pA = pN$, and therefore every point in the circumference of ANB is equidistant from P and p.

It is also obvious that P and p are the poles of every circle made by planes which are parallel to ANB. Hence they are the poles of the great circle DFE.

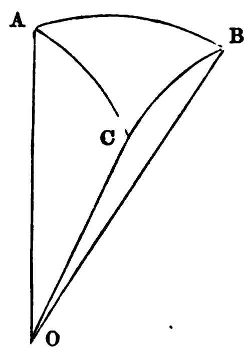
Cor. 1. Hence, since $PF = PD$, each being a quadrant, the distance of every point of a great circle from its pole is a quadrant.

Cor. Hence, since each of the planes POD, POF, is perpendicular to the plane DOF, the angles PDF, PFD, are right angles, and thence we may readily find the pole of any arc, as DF, by drawing from D and F two quadrantal arcs of great circles, each at right angles to DF, and the point P of their intersection will be the pole of DF.

9. A spherical triangle is a portion of the surface of a sphere contained by three arcs of three great circles, each of which is supposed to be less than a semicircle. The angles formed by the arcs are the angles of inclination of the planes to each other.

A spherical triangle is called quadrantal, if one side be a quadrant; and equilateral, isosceles and right-angled, in the same cases as in rectilinear triangles.

10. In every spherical triangle, any side is less than the sum of the other two.



ABC the spherical triangle.

Take *O* the centre of the sphere.

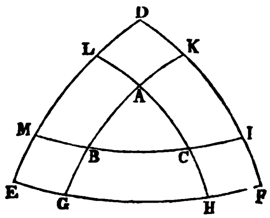
Draw the radii *OA*, *OB*, *OC*; then we may conceive that at *O*, there is formed a solid angle contained by the three plane angles *AOC*, *AOB*, *COB*, any two of which are greater than the third, and these are measured by the sides *AB*, *AC*, *BC*; hence any two sides of a spherical triangle are together greater than the remaining side.

11. The sum of the three sides of a spherical triangle is less than the circumference of a great circle.

For the solid angle at *O* is contained by plane angles, the sum of which is always less than four right angles. Hence the sum of their measures, which are the sides of the triangle, is always less than the circumference of a great circle.

Cor. Hence it is obvious that the sum of the sides of a spherical polygon is less than the circumference of a great circle.

The Polar Triangle.



12. Let ABC be a spherical triangle. With points, A, B, C, as poles, describe the arcs EF, DF, DE, forming the triangle DEF.

Then,

(1.) D, E, and F, shall be the poles of BC, AC, and AB respectively.

For $\because$ A is the pole of EF, AE is a quadrant.

And $\because$ C is the pole of DE, EC is a quadrant.

Hence A and C are each a quadrant distant from E.

$$\therefore \text{ E is the pole of } AC.$$

Similarly, F is the pole of AB, and D of BC.

(2.) The sides EF, FD, DE, are supplemental of the angles A, B, C.

Produce the sides, AB, AC, to meet EF in G and H.

Then $\because$ AG and AH are each quadrants; GH is the measure of the angle A.

Also $\because$ E is the pole of AC; EH is a quadrant, and similarly, FG is a quadrant.

Now, $GH=EH-EG=\frac{\pi}{2}-(FE-FG)=\frac{\pi}{2}-EF+\frac{\pi}{2}=\pi-EF$;

$$\therefore EF=\pi-GH=\pi-A.$$

In the same manner,

$$DF=\pi-B, \text{ and } DE=\pi-C.$$

(3.) The sides BC, AC, AB, are supplemental of the angles D, E, F.

For $BC = MC - MB = \frac{\pi}{2} - (MI - BI) = \frac{\pi}{2} - MI + \frac{\pi}{2} = \pi - MI$;

$\therefore$ MI, which measures D, $= \pi - BC$.

And similarly, $E = \pi - AC$, and $F = \pi - AB$.

Hence,

If A, B, C, a, b, c, represent the angles and sides of ABC,
and $A_{,}$, $B_{,}$, $C_{,}$, $a_{,}$, $b_{,}$, $c_{,}$, ... of DEF,

we have the following equations:

$$A = \pi - a_{,}, \text{ and } A_{,} = \pi - a.$$
$$B = \pi - b_{,}, \quad ,, \quad B_{,} = \pi - b.$$
$$C = \pi - c_{,}, \quad ,, \quad C_{,} = \pi - c.$$

These properties of the polar triangle will be found to be very useful in our succeeding investigations.

13. The sum of all the angles of a spherical triangle is less than six, and greater than two, right angles.

For A, B, C, being the angles of the triangle, and $a_{,}$, $b_{,}$, $c_{,}$, the sides of the corresponding polar triangle,

$$A + B + C = 3\pi - (a_{,} + b_{,} + c_{,});$$

$\therefore$ it is manifest that $A + B + C$ is $\angle\ 3\pi$.

But in every spherical triangle the sum of the sides is $\angle\ 2\pi$.

$$\therefore\ a_{,} + b_{,} + c_{,} \text{ is } \angle\ 2\pi.$$
$$\therefore\ A + B + C \text{ is } 7\ \pi.$$

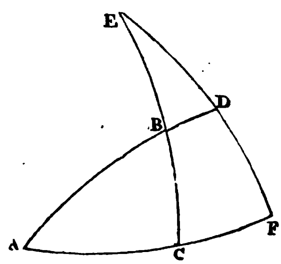
Cor. Hence, although the sum of the angles of a spherical triangle is not a definite quantity, yet it is contained between the limits of two and six right angles.

The Complemental Triangle.

14. Let ABC be a right-angled triangle, C being the right angle.

Produce AB to D, and CB to E, making AD and CE quadrants.

Join ED by the arc of a great circle, and produce it to meet AC, produced, in F.



Then the triangle EBD is called, from certain properties, the complemental triangle.

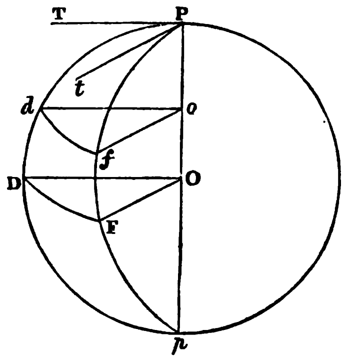
For $\because$ $EC = 90$, and ECF is a right angle; E is the pole of ACF. And $\therefore$ $EA = EF = 90°$.

Also, AE and AD each being quadrants, A is the pole of EF; and $\therefore$ AF is a quadrant. Hence, DF measures $\angle A$, but ED is the complement of DF; $\therefore$ ED is the complement of A.

And E is measured by FC, the complement of AC; $\therefore$ complement of $E = AC$.

And BD is the complement of AB, and EB of BC, which are the properties that give the name of the complemental triangle to EBD. EBD is right-angled, for A being the pole of EF, ADE is a right angle.

15. The inclination of two great circles is the angle made by their tangents at the point of their intersection.



For, let PT and Pt be the tangents at the point P, where two great circles intersect. Then, since Pp is the intersection of their planes, and PT, Pt are both perpendicular to PO; $\therefore \angle\ TPt$ measures the inclination.

Also, $\because$ OD is parallel to PT, and OF to Pt; $\therefore \angle\ DOF$, or the arc DF, measures the inclination.

16. To compare df, an arc of a small circle made by a plane parallel to DOF, with DF, the corresponding arc of a great circle.

Let o be the centre of the small circle.

Join od, of.

Then $df : DF :: od : OD :: \sin. Pd : 1$,

$\qquad\qquad\qquad\qquad\qquad$ or $:: \cos. Dd : 1$.

COR. If the spherical figure represent the earth, and P and p its poles, Dd will be the latitude of a place in the circle of which df is an arc; and if the angle at P be one degree.

DF is a degree of longitude at the equator, and
df a degree in latitude Dd or l; if $Dd = l$.

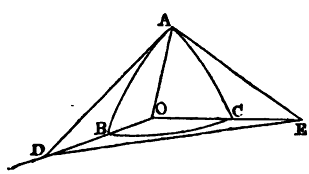
Hence, length of a degree of longitude at equator : length at latitude l :: 1 : cos. l.

CHAPTER II.

INVESTIGATION OF FORMULAS FOR THE COSINE AND SINE OF THE ANGLES AND SIDES OF A SPHERICAL TRIANGLE.

17. To find the cosine of the angle of a spherical triangle in terms of the cosines and sines of the sides.

Let O be the intersection of the three planes AOB, AOC, BOC. With centre O, and radius $OA = r$, describe three arcs, AB, BC, AC, forming the spherical triangle ABC.



From A draw AD, AE, tangents to AB and AC respectively, and terminated by OD, OE, drawn from O, through the other extremities B and C, of AB and AC. Join DE.

Let the $\angle^s$ be A, B, C, and the sides opposite a, b, c.

$$\therefore DE^2 = DA^2 + AE^2 - 2DA \,.\, AE \,.\, \cos. DAE \ldots (1.)$$
$$= DO^2 + OE^2 - 2DO \,.\, OE \,.\, \cos. BOC.$$

Whence

$$\because \angle DAE = A; \ \angle BOC = BC = a;$$

and $\angle^s DAO$, EAO are right angles,

$$\therefore OD^2 - AD^2 = OA^2 = OE^2 - AE^2;$$

$$\therefore AD \,.\, AE \,.\, \cos. A + OA^2 = OD \,.\, OE \,.\, \cos. a.$$

But $AD = OA \,.\, \tan. c$; $OD = OA \,.\, \sec. c = \dfrac{OA}{\cos. c}$;

$\therefore$ substituting and dividing by OA^2,

$$\therefore \tan. c \,.\, \tan. b \,.\, \cos. A + 1 = \frac{\cos. a}{\cos. b \,.\, \cos. c};$$

$$\therefore \cos. A = \frac{\cos. a - \cos. b \,.\, \cos. c}{\sin. b \,.\, \sin. c}.$$

By using a similar construction, and by the same reasoning, we might find the cosines of the other angles; but we may arrive at their values by mere substitution.

Thus, putting B for A, b for a, and a for b;

$$\cos. B = \frac{\cos. b - \cos. a \,.\, \cos. c}{\sin. a \,.\, \sin. c};$$

and similarly, $\cos. C = \dfrac{\cos. c - \cos. a \,.\, \cos. b}{\sin. a \,.\, \sin. b};$

18. To find the cosines of the sides of a spherical triangle in terms of the sines and cosines of the angles.

The demonstration of the preceding article being true for any triangle, holds true for the polar triangle.

Let $A_{,} B_{,} C_{,}$ be the $\angle^s$, and $a_{,} b_{,} c_{,}$ the sides of the polar triangle;

$$\therefore \cos. A_{,} = \frac{\cos. a_{,} - \cos. b_{,} \,.\, \cos. c_{,}}{\sin. b_{,} \,.\, \sin. c_{,}}.$$

But $A_{,} = \pi - a$, $b_{,} = \pi - B$, $c_{,} = \pi - C$, $a_{,} = \pi - A$;

$\therefore \cos. A_{,} = \cos. (\pi - a) = - \cos. a$; $\sin. b_{,} = \sin. \overline{\pi - B} = \sin. B$;

$$\therefore - \cos. a = \frac{- \cos. A - \cos. B \,.\, \cos. C}{\sin. B \,.\, \sin. C};$$

$$\therefore \cos. a = \frac{\cos. A + \cos. B \,.\, \cos. C}{\sin. B \,.\, \sin. C}.$$

Similarly, $\cos. b = \dfrac{\cos. B + \cos. A \,.\, \cos. C}{\sin. A \,.\, \sin. C};$

and $\cos. c = \dfrac{\cos. C + \cos. A \,.\, \cos. B}{\sin. A \,.\, \sin. B}.$

19. To find the sine of the angle of a spherical triangle in terms of the sines of the sides.

Since $\sin.^2 A = 1 - \cos.^2 A = (1 - \cos. A) . (1 + \cos. A)$;

$$\text{and } 1 + \cos. A = 1 + \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c}$$

$$= \frac{\cos. a - (\cos. b . \cos. c - \sin. b . \sin. c)}{\sin. b . \sin. c}$$

$$= \frac{\cos. a - \cos. (b + c)}{\sin. b . \sin. c} = \frac{2 \sin. \left(\frac{a+b+c}{2}\right) . \sin. \left(\frac{b+c-a}{2}\right)}{\sin. b . \sin. c}.$$

$$\text{Also, } 1 - \cos. A = 1 - \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c}$$

$$= \frac{\cos. b . \cos. c + \sin. b . \sin. c - \cos. a}{\sin. b . \sin. c}$$

$$= \frac{\cos. (b - c) - \cos a}{\sin. b . \sin. c} = \frac{2 \sin. \left(\frac{b+a-c}{2}\right) . \sin. \left(\frac{a+c-b}{2}\right)}{\sin. b . \sin. c}.$$

Make $b + a + c = 2s$;

$$\therefore a + b - c = 2s - 2c = 2(s - c);$$

$$a + c - b \qquad = 2(s - b).$$

$$\text{Whence } 1 + \cos. A = \frac{2 \sin. s . \sin. (s - a)}{\sin. b . \sin. c};$$

$$\text{and } 1 - \cos. A = \frac{2 \sin. (s - b) . \sin. (s - c)}{\sin. b . \sin. c};$$

$$\therefore \sin.^2 A = \frac{4 \{ \sin. s . \sin. (s - a) . \sin. (s - b) . \sin. (s - c) \}}{(\sin. b . \sin. c)^2};$$

$$\therefore \sin. A = \frac{2}{\sin. b . \sin. c} \sqrt{\sin. s . \sin. (s - a) . \sin. (s - b) . \sin. (s - c.)}$$

20. Sin. B may be deduced from sin. A, by changing A into B, a into b, and b into a; and similarly may we find sin. C; but as it is plain that the part under the square root does not alter its value when these changes are made, we may write, if we put $M = \sqrt{\sin. s \,.\, \sin.(s-a) \,.\, \sin.(s-b) \,.\, \sin.(s-c)}$,

$$\sin. A = \frac{2M}{\sin. b \,.\, \sin. c};$$

$$\sin. B = \frac{2M}{\sin. a \,.\, \sin. c};$$

$$\sin. C = \frac{2M}{\sin. a \,.\, \sin. b}.$$

21. Hence $\dfrac{\sin. A}{\sin. B} = \dfrac{\sin. a \,.\, \sin. c}{\sin. b \,.\, \sin. c} = \dfrac{\sin. a}{\sin. b};$

and $\dfrac{\sin. A}{\sin. C} = \dfrac{\sin. a \,.\, \sin. b}{\sin. b \,.\, \sin. c} = \dfrac{\sin. a}{\sin. c};$

or the *sines* of the *angles* are proportional to the *sines* of the *sides* which subtend them.

22. To find the sine of the side of a spherical triangle in terms of the angles.

Using the polar triangle, we have

$$\sin. A_{,} = \frac{2M_{,}}{\sin. b_{,} \,.\, \sin. c_{,}}.$$

$$\text{Whence } \sin. a = \frac{2M_{,}}{\sin. B \,.\, \sin. C};$$

since $A_{,} = \pi - a$, and $b_{,} = \pi - B$.

But $M_{,} = \sqrt{\sin. s_{,} \,.\, \sin.(s_{,} - a_{,}) \,.\, \sin.(s_{,} - b_{,}) \,.\, \sin.(s_{,} - c_{,})}$.

$$\text{Make } 2S = A + B + C.$$

$$\text{Now } s_{,} = \tfrac{1}{2}(a_{,} - b_{,} + c_{,}) = \tfrac{1}{2}(3\pi - \overline{A+B+C}) = \tfrac{1}{2}(3\pi - 2S),$$

$$\text{and } s_{,} - a_{,} = \frac{3\pi}{2} - S - \pi + A = \frac{\pi}{2} - \overline{S - A};$$

$$\therefore \sin. s_{,} = \sin. \left(\frac{3\pi}{2} - S\right) = -\cos. S;$$

$$\sin. (s_{,} - a_{,}) = \sin. \left(\frac{\pi}{2} - S - A\right) = \cos. (S - A);$$

$$\therefore M_{,} = \sqrt{-\cos. S \,.\, \cos. (S-A) \,.\, \cos. (S-B) \,.\, \cos. (S-C)};$$

$$\therefore \sin. a = \frac{2\sqrt{-\cos. S \,.\, \cos. (S-A) \,.\, \cos. (S-B) \,.\, \cos. (S-C)}}{\sin. B \,.\, \sin. C};$$

and from this expression sin. b and sin. c may be deduced.

23. The product of the factors under the sign of the square root appears to be a negative quantity; but it is really positive.

For since the sum of the angles of a spherical triangle is greater than two right angles, but less than six, S is always greater than $90°$, and less than $270°$; and therefore cos. S is negative.

Also, since in every spherical triangle, and therefore in the polar triangle, any two sides are greater than the third side,

$$\therefore a_{,} + b_{,} > c_{,};$$

$$\text{i. e. } \pi - A + \pi - B > \pi - C, \text{ or } A + B - C < \pi;$$

$$\therefore \frac{A + B - C}{2}, \text{ or } S - C, \text{ is } < \frac{\pi}{2};$$

$$\text{and } \therefore \cos. (S - C) \text{ is positive.}$$

Hence the product of the factors is a positive quantity.

17/

24. If in Art. ~~16~~ we make $a = b$,

$$\text{Then } \cos. A = \frac{\cos. a - \cos. a . \cos. c}{\sin. a . \sin. c},$$

$$\text{and } \cos. B = \frac{\cos. a - \cos. a . \cos. c}{\sin. a . \sin. c};$$

$$\therefore \cos. A = \cos. B; \quad \therefore A = B;$$

or the angles of isosceles triangles are also equal.

25. Also, in Art. 17. If we make $A = B$;

$$\text{then } \cos. a = \frac{\cos. A + \cos. A . \cos. C}{\sin. A . \sin. C},$$

$$\text{and } \cos. b = \frac{\cos. A + \cos. A . \cos. C}{\sin. A . \sin. C};$$

$$\therefore \cos. a = \cos. b, \text{ and } a = b.$$

Hence, if two angles of a spherical triangle be equal, the sides which subtend the equal angles are also equal.

26. To prove that

$$(1.) \ \text{Tan.} \left(\frac{A+B}{2}\right) = \frac{\cos. \left(\frac{a-b}{2}\right)}{\cos. \left(\frac{a+b}{2}\right)} . \cot. \frac{C}{2}.$$

$$(2.) \ \text{Tan.} \left(\frac{A-B}{2}\right) = \frac{\sin. \left(\frac{a-b}{2}\right)}{\sin. \left(\frac{a+b}{2}\right)} . \cot. \frac{C}{2}.$$

$$\because \cos. A = \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c};$$

$$\text{and } \cos. C = \frac{\cos. c - \cos. a . \cos. b}{\sin. a . \sin. b};$$

$$\therefore \cos. c = \cos. a . \cos. b + \sin. a . \sin. b . \cos. C;$$

$$\therefore \cos.A = \frac{\cos.a - \cos.a \cdot \cos.^2 b - \sin.a \cdot \sin.b \cdot \cos.b \cdot \cos.C}{\sin.b \cdot \sin.c}$$

$$= \frac{\cos.a \cdot (1-\cos.^2 b) - \sin.a \cdot \sin.b \cdot \cos.b \cdot \cos.C}{\sin.b \cdot \sin.c}$$

$$= \frac{\cos.a \cdot \sin.b - \sin.a \cdot \cos.b \cdot \cos.C}{\sin.c};$$

$$\therefore \text{ also, } \cos.B = \frac{\cos.b \cdot \sin.a - \sin.b \cdot \cos.a \cdot \cos.C}{\sin.c};$$

$$\therefore \cos.A + \cos.B = \frac{\sin.(a+b) - \sin.(a+b) \cdot \cos.C}{\sin.c}$$

$$= \frac{\sin.(a+b) \cdot (1-\cos.C)}{\sin.c} = \frac{2\sin.(a+b) \cdot \sin.^2 \frac{C}{2}}{\sin.c}.$$

But $\sin.A = \sin.a \cdot \dfrac{\sin.C}{\sin.c}$, and $\sin.B = \sin.b \cdot \dfrac{\sin.C}{\sin.c}$;

$$\therefore \sin.A + \sin.B = (\sin.a + \sin.b) \cdot \frac{\sin.C}{\sin.c};$$

$$\therefore \frac{\sin.A + \sin.B}{\cos.A + \cos.B} = \frac{(\sin.a + \sin.b) \cdot \sin.C}{2\sin.(a+b) \cdot \sin.^2 \frac{C}{2}};$$

$$\text{or } \tan.\left(\frac{A+B}{2}\right) = \frac{2\sin.\left(\frac{a+b}{2}\right) \cdot \cos.\left(\frac{a-b}{2}\right) \cdot 2\sin.\frac{C}{2} \cdot \cos.\frac{C}{2}}{4 \cdot \sin.\left(\frac{a+b}{2}\right) \cdot \cos.\left(\frac{a+b}{2}\right) \cdot \sin.^2 \frac{C}{2}}$$

$$= \frac{\cos.\left(\frac{a-b}{2}\right)}{\cos.\left(\frac{a+b}{2}\right)} \cdot \cot.\frac{C}{2}. \quad (1.)$$

And $\dfrac{\sin.A - \sin.B}{\cos.A + \cos.B} = \dfrac{(\sin.a - \sin.b)\sin.C}{2\sin.(a+b) \cdot \sin.^2 \frac{C}{2}};$

$$\text{or } \tan.\left(\frac{A-B}{2}\right)=\frac{2\sin.\left(\frac{a-b}{2}\right).\cos.\left(\frac{a+b}{2}\right).2\sin.\frac{C}{2}.\cos.\frac{C}{2}}{4.\sin.\left(\frac{a+b}{2}\right).\cos.\left(\frac{a+b}{2}\right).\sin.^2\frac{C}{2}}$$

$$=\frac{\sin.\left(\frac{a-b}{2}\right)}{\sin.\left(\frac{a+b}{2}\right)}.\cot.\frac{C}{2}. \quad (2.)$$

27. To prove that

$$(1.)\ \text{Tan.}\left(\frac{a+b}{2}\right)=\frac{\cos.\left(\frac{A-B}{2}\right)}{\cos.\left(\frac{A+B}{2}\right)}.\tan.\frac{c}{2}.$$

$$(2.)\ \text{Tan.}\left(\frac{a-b}{2}\right)=\frac{\sin.\left(\frac{A-B}{2}\right)}{\sin.\left(\frac{A+B}{2}\right)}.\tan.\frac{c}{2}.$$

Using the polar triangle, the sides and angles of which are denoted by $a_{,}\, b_{,}\, c_{,}$, $A_{,}\, B_{,}\, C_{,}$, we have from the preceding article,

$$\text{Tan.}\left(\frac{A_{,}+B_{,}}{2}\right)=\frac{\cos.\left(\frac{a_{,}-b_{,}}{2}\right)}{\cos.\left(\frac{a_{,}+b_{,}}{2}\right)}.\cot.\frac{C_{,}}{2}.$$

$$\text{But } \frac{A_{,}+B_{,}}{2}=\frac{2\pi-(a+b)}{2}=\pi-\left(\frac{a+b}{2}\right);$$

$$\therefore\ \tan.\left(\frac{A_{,}+B_{,}}{2}\right)=-\tan.\left(\frac{a+b}{2}\right);$$

$$\cos.\left(\frac{a_{,}-b_{,}}{2}\right)=\cos.\left(\frac{\overline{\pi-A}-\overline{\pi-B}}{2}\right)=\cos.\left(\frac{B-A}{2}\right)=\cos.\left(\frac{A-B}{2}\right);$$

$$\cos.\left(\frac{a_{,}+b_{,}}{2}\right)=\cos.\left(\pi-\frac{A+B}{2}\right)=-\cos.\left(\frac{A+B}{2}\right).$$

$$\text{Cot.}\,\frac{C_{\prime}}{2} = \text{cot.}\left(\frac{\pi - c}{2}\right) = \text{tan.}\,\frac{c}{2}\,.$$

Whence, making the proper substitutions, and changing the signs on both sides,

$$\text{Tan.}\left(\frac{a+b}{2}\right) = \frac{\cos.\left(\frac{A-B}{2}\right)}{\cos.\left(\frac{A+B}{2}\right)}\,.\,\text{tan.}\,\frac{c}{2}. \quad (3.)$$

And after a similar manner we may prove, that

$$\text{Tan.}\left(\frac{a-b}{2}\right) = \frac{\sin.\left(\frac{A-B}{2}\right)}{\sin.\left(\frac{A+B}{2}\right)}\,.\,\text{tan.}\,\frac{c}{2}. \quad (4.)$$

The four formulas marked (1), (2), (3), (4), are called Napier's Analogies; the former two are used when two sides, and the angle included by them, are given; the latter two, when two sides, and the angle opposite the third side, are given, the third side being required.

CHAPTER III.

SOLUTION OF SPHERICAL TRIANGLES.

28. In a spherical triangle there are six parts, the three angles, and the three sides; and, from the preceding chapter, we may infer that, if three of these quantities be given, the rest may be found.

For, since

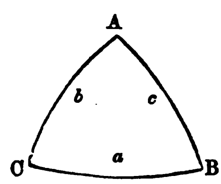
$$\cos. A = \frac{\cos. a - \cos. b . \cos. c}{\sin. b \,.\, \sin. c}, \ \& \cos. a = \frac{\cos. A + \cos. B . \cos. C}{\sin. B \,.\, \sin. C};$$

$$\text{and also, } \frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b};$$

it is obvious that any one of the parts of the triangle is dependent upon three others, and these three being given, the fourth may be found.

Right-angled Spherical Triangles.

29. We shall begin with right-angled spherical triangles, and suppose C to be the right angle.



We may, from the elementary formulas, obtain the solution of the triangle, when two of the quantities, a, b, c, A, B, are given; but, in practice, other rules easy of application are used.

These rules were invented by Napier, and are termed "Napier's Rules for the Circular Parts."

He called, the two sides (a), (b), the complement of the hypothenuse, or $90 - c$, the complements of the two angles, or $90 - A$ and $90 - B$, circular parts; and, any one of these being termed a middle part, the two nearest, one on each side, are named *adjacent* parts, and the other two, *extreme* parts. The rules are these:—

(1.) Sine of middle part = product of the tangents of the adjacent parts.

(2.) Sine of middle part = product of the cosines of the extreme parts.

Thus, let (a) be the middle part; $\therefore$ $90 - B$ and (b) are the adjacent, and $90 - c$ and $90 - A$ are the extreme parts.

$\therefore$ $\sin. a = \tan. (90 - B) . \tan. b = \cot. B . \tan. b$;
and $\sin. a = \cos. (90 - c) . \cos. (90 - A) = \sin. c . \sin. A$.

Again, let $(90 - B)$ be a middle part; then a and $90 - c$ are the adjacent, and $(90 - A)$ and b the extreme parts;

$$\therefore \sin. (90 - B) = \tan. a . \tan. (90 - c);$$
$$\therefore \cos. B = \tan. a . \cot. c,$$
$$\text{and } \sin. (90 - B) = \cos. (90 - A) . \cos. b;$$
$$\therefore \cos. B = \sin. A . \cos. b.$$

Now, let $(90 - c)$ be the middle part; then $\overline{90 - A}$ and $\overline{90 - B}$ are the adjacent parts, and a and b the extreme parts.

$$\therefore \sin. \overline{90 - c} = \tan. (90 - A) . \tan. \overline{90 - B};$$
$$\therefore \cos. c = \cot. A . \cot. B;$$
$$\text{and } \sin. \overline{90 - c} = \cos. a . \cos. b;$$
$$\therefore \cos. c = \cos. a . \cos. b.$$

We may also obtain four other formulas, two for sin. b, and two for cos. A, by first making (b) and then $\overline{90-A}$ the middle parts; but they are of the same form as those deduced for sin. a and cos. B, and the investigation is therefore needless.

30. To show the truth of these rules.

(1.) Since $C = 90$;

$$\therefore\ \cos. C = 0 = \frac{\cos. c - \cos. a \,.\, \cos. b}{\sin. a \,.\, \sin. b};$$

$$\therefore\ \cos. c = \cos. a \,.\, \cos. b. \quad (1.)$$

Also,

$$\cos. c = \frac{\cos. C + \cos. A \,.\, \cos. B}{\sin. A \,.\, \sin. B}.$$

$$= \frac{\cos. A \,.\, \cos. B}{\sin. A \,.\, \sin. B} = \cot. A \,.\, \cot. B; \quad (2.)$$

$$\therefore\ \sin.(90 - c) = \cos. a \,.\, \cos. b,$$
$$\text{and } \sin.(90 - c) = \tan.(90 - A) \,.\, \tan.(90 - B).$$

(2.) Again, $\cos. B = \dfrac{\cos. b - \cos. a \,.\, \cos. c}{\sin. a \,.\, \sin. c}$.

But $\cos. b = \dfrac{\cos. c}{\cos. a}$; $\therefore\ \cos. B = \dfrac{\dfrac{\cos. c}{\cos. a} - \cos. a \,.\, \cos. c}{\sin. a \,.\, \sin. c}$

$$= \frac{\cos. c \,.\, (1 - \cos.^2 a)}{\sin. a \,.\, \cos. a \,.\, \sin. c} = \frac{\cos. c \,.\, \sin. a}{\sin. c \,.\, \cos. a} = \cot. c \,.\, \tan. a. \quad (3.)$$

And $\cos. b = \dfrac{\cos. B + \cos. A \,.\, \cos. C}{\sin. A \,.\, \sin. C}$, and $\cos. C = 0$; $\sin. C = 1$;

$$\therefore\ \cos. b = \frac{\cos. B}{\sin. A};\ \therefore\ \cos. B = \cos. b \,.\, \sin. A. \quad (4.)$$

$$\therefore\ \sin.(90 - B) = \tan.(90 - c) \,.\, \tan. a;$$
$$\text{and } \sin.(90 - B) = \cos. b \,.\, \cos.(90 - A).$$

(3.) And $\sin. a = \dfrac{\sin. A \,.\, \sin. b}{\sin. B}$.

But, from (4) $\sin. A = \dfrac{\cos. B}{\cos. b}$;

$$\therefore \sin. a = \frac{\cos. B \,.\, \sin. b}{\sin. B \,.\, \cos. b} = \cot. B \,.\, \tan. b. \quad (5.)$$

$$\text{And } \sin. a = \frac{\sin. A \,.\, \sin. c}{\sin. C} = \sin. A \,.\, \sin. c. \quad (6.)$$

31. We shall now repeat these formulas, for the purpose of reference.

(1.) $\text{Cos. } c = \cos. a \,.\, \cos. b.$
(2.) $\text{Cos. } c = \cot. A \,.\, \cot. B.$
(3.) $\text{Cos. } B = \cot. c \,.\, \tan. a.$
(4.) $\text{Cos. } B = \cos. b \,.\, \sin. A.$
(5.) $\text{Sin. } a = \cot. B \,.\, \tan. b.$
(6.) $\text{Sin. } a = \sin. A \,.\, \sin. c.$

Hence also, by substitution,

(7.) $\text{Cos. } A = \cot. c \,.\, \tan. b.$
(8.) $\text{Cos. } A = \cos. a \,.\, \sin. B.$
(9.) $\text{Sin. } b = \cot. A \,.\, \tan. a.$
(10.) $\text{Sin. } b = \sin. B \,.\, \sin. c.$

32. It will be easily understood that, in the solutions of the triangles given by Napier's rules, cases of ambiguity will occur. These may happen whenever the required values are to be determined through the means of the sines of the arcs or angles.

Now, referring to Art. 31, it will be seen that, if A and a are given, c, B, and b, are, by the formulas

(6), (8), (9), to be computed from their sines; whence an ambiguity will arise, since we have no means of knowing whether an arc or its supplement is to be taken.

Again, if A and (c) be given, and (a) be required, we have, by formula, (6),

$$\sin. a = \sin. A \,.\, \sin. c;$$

and it would appear, at first sight, that there was an ambiguity similar to that which we have just mentioned, but the doubt is removed by reference to (9), whence we have

$$\tan. a = \sin. b \,.\, \tan. A.$$

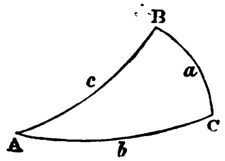
For, since $\sin. b$ is always positive, and since the tangent is positive or negative according as the arc is less or greater than 90°, $\tan. a$ will be positive or negative according as $\tan. A$ is positive or negative, or (a) will be less or greater than 90° according as A is less or greater than 90°.

If (a) and (c) be given to find A, we must reason in the same manner.

33. These ten formulas are sufficient to solve all cases of right-angled spherical triangles. For, as there are five quantities, a, b, c, A, and B, and these are taken three and three together, the number of different combinations which can arise does not exceed ten.

34. We may solve, by the same rules, the triangle in which one side is a quadrant; for the polar triangle will have the corresponding angle a right angle, and the sides and angles of the polar triangle being found, the angles and sides of the proposed triangle may also be found.

35. The following are examples of the solution of right-angled spherical triangles.



Ex. 1. Given $\angle A$, b, to find (c), (a), and $\angle B$.

Let $(90-A)$ be the middle part; $\therefore$ b and $\overline{90-c}$ are the adjacent parts.

$$\therefore \sin.(90-A) = \tan. b \,.\, \tan.(90-c);$$

$$\therefore \cos. A = \tan. b \,.\, \cot. c;$$

$$\therefore \cot. c = \frac{\cos. A}{\tan. b};$$

$$\therefore \log. \cot. c = \log. \cos. A - \log. \tan. b + 10,$$

whence (c)

Again, $\sin. b = \tan. a \,.\, \tan.(90-A) = \tan. a \,.\, \cot. A$;

$$\therefore \tan. a = \sin. b \,.\, \tan. A;$$

$$\therefore \log. \tan. a = \log. \sin. b + \log. \tan. A - 10,$$

whence (a).

Again, $90-B$ being the middle part, b and $90-A$ are the extreme parts;

$$\therefore \sin.(90-B) = \cos. b \,.\, \cos. \overline{90-A};$$

$$\therefore \cos. B = \cos. b \,.\, \sin. A;$$

$$\therefore \log. \cos. B = \log. \cos. b + \log. \sin. A - 10,$$

whence B.

Ex. 2. Given a, c; find b, A, B.

$\sin. \overline{90-c} = \cos. a \,.\, \cos. b$; $\therefore$ $\cos. c = \cos. a \,.\, \cos. b$;

$$\therefore \log. \cos. b = \log. \cos. c + 10 - \log. \cos. a.$$

Again, $\sin.(90-B) = \tan. a \,.\, \tan.(90-c)$;

$$\therefore \cos. B = \tan. a \,.\, \cot. c;$$

$$\therefore \log. \cos. B = \log. \tan. a + \log. \cot. c - 10.$$

$$\text{Since } \sin. A = \frac{\sin. a}{\sin. c};$$

$$\therefore \log. \sin. A = \log. \sin. a - \log. \sin. c + 10;$$

and by Art. 32, there is no ambiguity in this case.

Oblique-angled Spherical Triangles.

36. The solution of oblique-angled spherical triangles may be included in six cases.

Case (1.) Given the three sides, find the three angles.

By Art. 16,

$$\cos. A = \frac{\cos. a - \cos. b \,.\, \cos. c}{\sin. b \,.\, \sin. c}.$$

This formula is, however, not adapted to logarithms.

$$\text{But } 1 - \cos. A = \frac{\cos. b \,.\, \cos. c + \sin. b \,.\, \sin. c - \cos. a}{\sin. b \,.\, \sin. c}$$

$$= \frac{\cos. (b - c) - \cos. a}{\sin. b \,.\, \sin. c};$$

$$\text{or, } 2 \sin.^2 \frac{A}{2} = \frac{2 \sin. (s - b) \,.\, \sin. (s - c)}{\sin. b \,.\, \sin. c}. \quad (1)$$

Therefore,

$$\log. \sin. \frac{A}{2} = 10 + \tfrac{1}{2} \left\{ \log. \sin. (s - b) + \log. \sin. (s - c) - \log. \sin. b - \log. \sin. c \right\};$$

$$\text{and } 1 + \cos. A = \frac{\cos. a - \cos. (b + c)}{\sin. b \,.\, \sin. c};$$

$$\therefore \cos.^2 \frac{A}{2} = \frac{\sin. s \,.\, \sin. (s - a)}{\sin. b \,.\, \sin. c}. \quad (2.)$$

$\therefore$ dividing (1) by (2), and extracting the root,

$$\tan. \frac{A}{2} = \sqrt{\frac{\sin. (s - b) \,.\, \sin. (s - c)}{\sin. s \,.\, \sin. (s - a)}}. \quad (3.)$$

Formula (3) is the most convenient, if all the angles be required; all are adapted for logarithmic tables.

Case (2.) Given the three angles, find the three sides.

By the formula of Art. 17,

$$\cos. a = \frac{\cos. A + \cos. B . \cos. C}{\sin. B . \sin. C}.$$

and by adding unity we may obtain, as in the preceding case, a formula for $\cos. \frac{a}{2}$.

$$\text{Cos.} \frac{a}{2} = \sqrt{\frac{\cos. (S-B)'. \cos. (S-C)}{\sin. B . \sin. C}}.$$

Or, making $\cos. \theta = \cos. B . \cos. C$, whence θ may be found, we have,

$$\cos. a = \frac{\cos. A + \cos. \theta}{\sin. B . \sin. C} = \frac{2 \cos. \frac{A+\theta}{2} . \cos. \frac{A-\theta}{2}}{\sin. B . \sin. C}.$$

This is a case which seldom occurs.

Case (3.) Given two sides and the included angle, find the remaining angles and side.

Let a, b, and C be the given parts of the triangle.

Then, from Napier's Analogies,

$$\tan. \left(\frac{A+B}{2}\right) = \frac{\cos. \left(\frac{a-b}{2}\right)}{\cos. \left(\frac{a+b}{2}\right)} . \cot. \frac{C}{2};$$

$$\tan. \left(\frac{A-B}{2}\right) = \frac{\sin. \left(\frac{a-b}{2}\right)}{\sin. \left(\frac{a+b}{2}\right)} . \cot. \frac{C}{2};$$

whence $\frac{A+B}{2}$ and $\frac{A-B}{2}$ being computed, A and B may be found.

And to find (c), $\sin. c = \dfrac{\sin. C . \sin. a}{\sin. A}$.

We may, however, find (c) independently, by means of a subsidiary angle.

$$\text{For, } \cos. c = \cos. a . \cos. b + \sin. b . \sin. a . \cos. C$$

$$= \cos. a . \cos. b + \sin. a . \sin. b . \left(2\cos.^2 \frac{C}{2} - 1\right)$$

$$= \cos. (a+b) + 2\sin. a . \sin. b . \cos.^2 \frac{C}{2}$$

$$= \cos. (a+b) \left\{ 1 + \frac{2\sin. a . \sin. b . \cos.^2 \frac{C}{2}}{\cos. (a+b)} \right\}.$$

$$\text{Let } \tan.^2 \theta = \frac{2\sin. a . \sin. b . \cos.^2 \frac{C}{2}}{\cos. (a+b)};$$

$$\therefore \cos. c = \cos. (a+b) \{1 + \tan.^2 \theta\} = \frac{\cos. (a+b)}{\cos.^2 \theta};$$

$$\log. \cos. c = \log. \cos. (a+b) + 20 - 2 \log. \cos. \theta;$$

$$\text{or, } \cos. c = \cos. a . \cos. b + \sin. a . \sin. b . \left(1 - 2\sin.^2 \frac{C}{2}\right)$$

$$= \cos. (a-b) - 2\sin. a . \sin. b . \sin.^2 \frac{C}{2}$$

$$= \cos. (a-b) \left\{ 1 - \frac{2\sin. a . \sin. b . \sin.^2 \frac{C}{2}}{\cos. (a-b)} \right\}$$

$$= \cos. (a-b) \{1 - \sin.^2 \theta\}$$

$$= \cos. (a-b) . \cos.^2 \theta;$$

$$\therefore \log. \cos. c = \log. \cos. (a-b) + 2\{\log. \cos. \theta - 10\};$$

where θ is to be found from the equation,

$$\sin.^2 \theta = \frac{2\sin. a . \sin. b . \sin.^2 \frac{C}{2}}{\cos. (a-b)}.$$

Or (c) may be thus found.

$$\because \cos. c = \cos. a . \cos. b + \sin. a . \sin. b . \cos. C$$

$$= \cos. (a + b) + 2 \sin. a . \sin. b . \cos.^2 \frac{C}{2};$$

$$\therefore 1 - \cos. c = 1 - \cos. (a+b) - 2 \sin. a . \sin. b . \cos.^2 \frac{C}{2};$$

$$\text{or, } \sin.^2 \frac{c}{2} = \sin.^2 \left(\frac{a+b}{2}\right) - \sin. a . \sin. b . \cos.^2 \frac{C}{2};$$

$$\text{Let } \sin. a . \sin. b . \cos.^2 \frac{C}{2} = \sin.^2 \theta;$$

$$\therefore \sin.^2 \frac{c}{2} = \sin.^2 \frac{a+b}{2} - \sin.^2 \theta$$

$$= \sin. \left(\frac{a+b}{2} + \theta\right) . \sin. \left(\frac{a+b}{2} - \theta\right).$$

And θ may be found from

$$\sin. \theta = \cos. \frac{C}{2} \sqrt{\sin. a . \sin. b}.$$

Case (4.) Given two angles, and a side opposite the other angle; find the remaining angle and sides.

Let A, B, and c, be given, and C, a, b, required.
Then, from Napier's Analogies, (3) and (4),

$$\tan. \left(\frac{a+b}{2}\right) = \frac{\cos. \left(\frac{A-B}{2}\right)}{\cos. \left(\frac{A+B}{2}\right)} . \tan. \frac{c}{2};$$

$$\tan. \left(\frac{a-b}{2}\right) = \frac{\sin. \left(\frac{A-B}{2}\right)}{\sin. \left(\frac{A+B}{2}\right)} \tan. \frac{c}{2}.$$

We may find $\frac{a+b}{2}$ and $\frac{a-b}{2}$, and therefore have,

$$a = \frac{a+b}{2} + \frac{a-b}{2}, \text{ and } b = \frac{a+b}{2} - \frac{a-b}{2}.$$

And C may be found from $\sin. C = \sin. c . \frac{\sin. A}{\sin. a}$;

or it may be computed from the formula,

$$\cos. C = \cos. c . \sin. A . \sin. B - \cos. A . \cos. B.$$

Case (5.) Given two sides, and an angle opposite one of them; find the remaining side and angles.

Let the given sides and angle be a, b, A.

$$\therefore \sin. B = \frac{\sin. b . \sin. A}{\sin. a}, \text{ whence } B.$$

To find (c).

$$\text{Cos.}\, a = \cos. b . \cos. c + \sin. b . \sin. c . \cos. A$$

$$= \cos. b \left\{ \cos. c + \frac{\sin. b . \cos. A}{\cos. b} . \sin. c \right\}.$$

Let $\cot. \theta = \tan. b . \cos. A$;

$$\therefore \cos. a = \cos. b \{ \cos. c + \cot. \theta . \sin. c \}$$

$$= \frac{\cos. b \{ \sin. \theta . \cos. c + \cos. \theta . \sin. c \}}{\sin. \theta}$$

$$= \frac{\cos. b . \sin. \overline{\theta + c}}{\sin. \theta}.$$

And θ being first computed, $(\theta + c)$ may be found, and therefore (c); and then $\sin. C = \sin. c . \frac{\sin. A}{\sin. a}$ may be found.

But C may be found independently of the value of (c).

For $\cos. a = \dfrac{\cos. A + \cos. B . \cos. C}{\sin. B . \sin. C}$;

$\therefore \cos. a . \sin. B . \sin. C = \cos. A + \cos. B . \cos. C.$

But, $\sin. B = \dfrac{\sin. A . \sin. b}{\sin. a}$,

and $\cos. B = \sin. A . \sin. C, \cos. b - \cos. A . \cos. C$;

Therefore, $\cot. a . \sin. b . \sin. A . \sin. C$

$= \cos. A + \cos. C (\sin. A . \sin. C . \cos. b - \cos. A . \cos. C)$

$= \cos. A . \sin.^2 C + \sin. A . \sin. C . \cos. C . \cos. b.$

Divide each side by $\sin. A . \sin. C$;

$$\therefore \cot. a . \sin. b = \cot. A . \sin. C + \cos. C . \cos. b$$
$$= \cot. A \{\sin. C + \cos. C . \cos. b . \tan. A\}$$

Let $\tan. \theta = \tan. A . \cos. b$;

$$\therefore \cot. a . \sin. b = \cot. A \{\sin. C + \tan. \theta . \cos. C\}$$
$$= \frac{\cot. A . \sin. \overline{(C + \theta)}}{\cos. \theta}$$
$$= \frac{\cos. b . \sin. (C + \theta)}{\sin. \theta};$$

$$\therefore \sin. (C + \theta) = \frac{\sin. \theta . \tan. b}{\tan. a};$$

whence $C + \theta$ may be found, when θ is computed.

Case (6.) Given two angles and a side opposite to one of them; find the remaining sides and angle.

Let A, B, and (a), be the given quantities.

Then, $\sin. b = \dfrac{\sin. B}{\sin. A} . \sin. a$, whence ($b$) may be found; and ($c$) may be found when ($b$) is known, as in the preceding case, or independently, thus:

$$\sin. b . \sin. c . \cos. A = \cos. a - \cos. b . \cos. c.$$

But $\sin. b = \dfrac{\sin. B . \sin. a}{\sin. A}$,

and $\cos. b = \sin. a . \sin. c . \cos. B + \cos. a . \cos. c$;

Therefore, $\sin. B . \sin. a . \sin. c . \cot. A$

$$= \cos. a - \cos. c \{ \sin. a . \sin. c . \cos. B + \cos. a . \cos. c \}$$

$$= \cos. a . \sin.^2 c - \cos. c . \sin. a . \sin. c . \cos. B.$$

Dividing by $\sin. a . \sin. c$;

$$\therefore \sin. B . \cot. A = \cot. a . \sin. c - \cos. c . \cos. B$$

$$= \cot. a \{ \sin. c - \cos. c . \cos. B . \tan. a \}$$

Let $\cos. B . \tan. a = \tan. \phi$;

$$\therefore \sin. B . \cot. A = \frac{\cot. a \{ \sin. (c - \phi) \}}{\cos. \phi} = \frac{\cos. B . \sin. (c - \phi)}{\sin. \phi};$$

$$\therefore \sin. (c - \phi) = \sin. \phi \frac{\tan. B}{\tan. A}, \text{ whence } c.$$

And C may be found from $\sin. C = \dfrac{\sin. A}{\sin. a} . \sin. c$, or in terms of A, B, and (a), from the equation,

$$\cos. A = \sin. B . \sin. C . \cos. a - \cos. B . \cos. C$$

$$= \cos. B \{ \tan. B . \cos. a . \sin. c - \cos. C \}.$$

Make $\tan. B . \cos. a = \cot. \theta$;

$$\therefore \cos. A = \cos. B \left\{ \frac{\sin. (C - \theta)}{\sin. \theta} \right\};$$

$$\therefore \sin.(C-\theta) = \sin.\theta \cdot \frac{\cos. A}{\cos. B}.$$

37. The six cases here solved contain all that occur in the solution of spherical triangles. The examples which illustrate them are chiefly to be found in the Treatises on Plane Astronomy, and in the Accounts of Trigonometrical Surveys; and to these the reader must be referred. In perusing the descriptions of Trigonometrical Surveys, he will find that, although for small distances the earth may be considered to be a plane, and therefore the rules of the first portion of this Treatise are sufficient, the spherical form of the earth cannot be neglected, when the magnitude of the operation exceeds certain limits, and then he will recognise the utility of the formulas, the truth of which has just been established.

EXAMPLES.

(1.) If c be the hypothenuse of a right-angled triangle,

$$[1.]\ \text{Sin.}^2\frac{c}{2} = \sin.^2\frac{a}{2} \cdot \cos.^2\frac{b}{2} + \cos.^2\frac{a}{2}\sin.^2\frac{b}{2}.$$

$$[2.]\ \text{Tan.}\left(\frac{c+a}{2}.\right)\tan.\left(\frac{c-a}{2}\right) = \left(\tan.\frac{b}{2}\right)^2.$$

(2.) In an oblique-angled triangle, if $a + b = \pi$, then $\sin.2A + \sin.2B = 0$.

(3.) In every triangle show that

$$\cos.a + \cos.b = \frac{2\sin.(A+B)}{\sin.C}\left(\cos.\frac{c}{2}\right)^2.$$

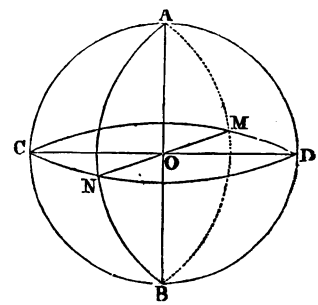
(4.) If $CD = D$, be drawn bisecting AB in D,

$$\cos.D = \frac{\sin.(A+B)}{\sin.C} \cdot \cos.\frac{c}{2}.$$

L

CHAPTER IV.

AREA OF LUNE. AREA OF SPHERICAL TRIANGLE, IN TERMS OF THE ANGLES AND OF THE SIDES.



38. Definition. The spherical surface, $ACBNA$, included between two semicircles, ACB, ANB, is called a lune.

39. Proposition. The area of the lune is proportional to the angle CAN.

For if CN, or angle CAN, increase, the area of the lune increases; and whatever multiple of CN be taken, the same multiple will the increased lune be of the lune $CANB$. Hence the lune is measured by CN. But, if CN become the whole circumference, the lune becomes the spherical surface.

And, therefore,

$$\text{lune} : \text{spherical surface } (S) :: CN : \text{circumference},$$
$$:: \angle CAN : 2\pi.$$

Let $\angle CAN = \theta$, (θ being expressed in parts of the radius unity.)

$$\therefore \text{lune} = S \cdot \frac{\theta}{2\pi}$$

But $S = 4\pi r^2$, (for so it is found to be, r being radius.)

$$\therefore \text{lune} \doteq r^2 \,.\, 2\theta, \text{ or } \propto 2\theta.$$

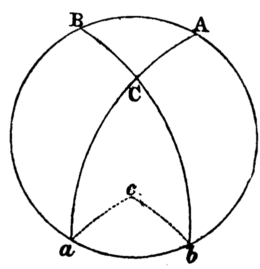
EXAMPLE. The angle of a lune being 30°, find the surface.

$$\text{Here, } 2\theta : 3.14159 :: 60 : 180 :: 1 : 3;$$

$$\therefore 2\theta = \frac{3.14159}{3} = 1.04719;$$

$$\therefore \text{area} = r^2 (1.04719).$$

40. To find the area of a spherical triangle, in terms of its angles.



Let ABC be the spherical triangle, and let the sides be produced till they meet.

In the figure, BC and AC are supposed to meet upon the other side of the sphere at c; and a triangle (abc) is formed, which we shall show to be equal to ABC.

For Aa, and Bb, each being semicircles, $\therefore AB = ab$; and $\because Aa = Cac$, $\therefore AC = ac$; and $\because Bb = Cbc$, $\therefore BC = bc$; and the angle at $C = \angle$ at c, and the remaining angles are equal; $\therefore$ the $\Delta\, ABC = \Delta\, abc$.

Let area of $\Delta\, ABC = x$, $\quad \Delta\, ACb = Q$,
. . . . $\Delta\, BCa = P$, $\quad \Delta\, Cab = R$.

Then, by the preceding proposition, ABC being the angles of the triangle,

$$x + P = 2r^2 A,$$
$$x + Q = 2r^2 B,$$
$$x + R = 2r^2 C;$$

$$\therefore 2x + (x + P + Q + R) = 2r^2 (A + B + C).$$

$$\text{But } x + P + Q + R = \text{hemisphere} = 2\pi r^2;$$
$$\therefore\ x = r^2 (A + B + C - \pi).$$

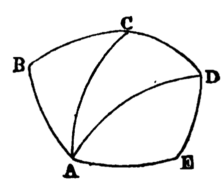
COR. 1. Hence, if $r = 1$, the area $= A + B + C - \pi$;
$$\therefore\ \text{area} \propto A^\circ + B^\circ + C^\circ - 180^\circ.$$

COR. 2. Hence the area of a spherical triangle is proportional to the excess of the sum of its angles above two right angles. This is commonly called the *spherical excess*.

Ex. Let $A = 90^\circ$, $B = 80^\circ$, $C = 70^\circ$; find the area of the triangle.

$$\text{Here } A + B + C - 180^\circ = 240^\circ - 180^\circ = 60^\circ = \frac{180^\circ}{3};$$
$$\therefore\ \text{area} = r^2 \frac{3.14159}{3} = r^2 (1.04719).$$

41. Hence to find the area of a spherical polygon of (n) sides.



For, by drawing arcs from a given point, A, to the angles of the figure, the polygon may be divided into $(n - 2)$ triangles; and as the area of each triangle is equal to r^2 {sum of its angles $- \pi$}, therefore the area of the polygon, which is the same as the sum of the areas of the triangles, will equal r^2 {sum of all the angles of the triangles $- (n - 2)\pi$}.

But the sum of all the angles of these triangles = sum of all the angles of the figure.

$= \Sigma (A)$, $\Sigma (A)$ expressing the sum of all the angles;

$$\therefore\ \text{surface of spherical polygon} = r^2 \{\Sigma (A) - \overline{n - 2}\,\pi\}.$$

42. To find the area of a spherical triangle in terms of two sides and the included angle.

Let a, b be the sides, and C the included angle; and let $S = A + B + C - 180$.

To prove that $\cot.\frac{S}{2} = \dfrac{\cot.\frac{a}{2}.\cot.\frac{b}{2} + \cos.C}{\sin.C}$.

For $\cot.\frac{S}{2} = \cot.\left(\frac{A+B+C}{2} - 90\right) = -\tan.\left(\frac{A+B+C}{2}\right)$

$$= -\frac{\tan.\left(\frac{A+B}{2}\right) + \tan.\frac{C}{2}}{1-\tan.\left(\frac{A+B}{2}\right).\tan.\frac{C}{2}}.$$

But $\tan.\left(\frac{A+B}{2}\right) = \dfrac{\cos.\left(\frac{a-b}{2}\right)}{\cos.\left(\frac{a+b}{2}\right)}.\cot.\frac{C}{2}$;

$$\therefore \cot.\frac{S}{2} = \frac{\cos.\left(\frac{a-b}{2}\right).\cot.\frac{C}{2} + \cos.\left(\frac{a+b}{2}\right).\tan.\frac{C}{2}}{\cos.\left(\frac{a-b}{2}\right) - \cos.\left(\frac{a+b}{2}\right)}$$

$$= \frac{\cos.\frac{a}{2}.\cos.\frac{b}{2}.\left(\cot.\frac{C}{2} + \tan.\frac{C}{2}\right) + \sin.\frac{a}{2}.\sin.\frac{b}{2}.\left(\cot.\frac{C}{2} - \tan.\frac{C}{2}\right)}{2\sin.\frac{a}{2}.\sin.\frac{b}{2}}$$

$$= \cot.\frac{a}{2}.\cot.\frac{b}{2}.\frac{1}{\sin.C} + \frac{\cos.C}{\sin.C}$$

$$= \frac{\cot.\frac{a}{2}.\cot.\frac{b}{2} + \cos.C}{\sin.C},$$

43. To find the area of a spherical triangle in terms of the sides.

$$\because \cot.\frac{S}{2} = \frac{\cot.\frac{a}{2}.\cot.\frac{b}{2} + \cos.C}{\sin.C}.$$

$$\text{Now, } \cos.C = \frac{\cos.c - \cos.a.\cos.b}{\sin.a.\sin.b}.$$

And $\cot.\frac{a}{2}.\cot.\frac{b}{2}$

$$= \frac{2\cos.^2\frac{a}{2}}{2\sin.\frac{a}{2}.\cos.\frac{a}{2}} \times \frac{2\cos.^2\frac{b}{2}}{2\sin.\frac{b}{2}.\cos.\frac{b}{2}} = \frac{1+\cos.a}{\sin.a}.\frac{1+\cos.b}{\sin.b};$$

$$\therefore \cot.\frac{a}{2}.\cot.\frac{b}{2} + \cos.C = \frac{1+\cos.a+\cos.b+\cos.c}{\sin.a.\sin.b}.$$

And

$$\sin.C = \frac{2}{\sin.a.\sin.b}\sqrt{\sin.(s).\sin.(s-a).\sin.(s-b).\sin.(s-c)};$$

$$\therefore \cot.\frac{S}{2} = \frac{1+\cos.a+\cos.b+\cos.c}{2\sqrt{\sin.(s).\sin.(s-a).\sin.(s-b).\sin.(-c)}}$$

an expression for the area in terms of the sides, but which is not adapted to logarithmic computation.

44. Next, to find an expression for the area of a spherical triangle, which shall be adapted to logarithmic computation.

$$\because \cot.\frac{S}{2} = \frac{\cot.\frac{a}{2}.\cot.\frac{b}{2} + \cos.C}{\sin.C};$$

$$\therefore\ 1+\cot.^2\frac{S}{2}=\frac{1}{\sin.^2\frac{S}{2}}$$

$$=\frac{\cot.^2\frac{a}{2}\,.\,\cot.^2\frac{b}{2}+2\cos.C\,.\,\cot.\frac{a}{2}\,.\,\cot.\frac{b}{2}+1}{\sin.^2C}.$$

But
$$2\cos.C\,.\,\cot.\frac{a}{2}\,.\,\cot.\frac{b}{2}=\frac{\cos.c-\cos.a\,.\,\cos.b}{2\sin.^2\frac{a}{2}\,.\,\sin.^2\frac{b}{2}}$$

$$=\frac{\sin.^2\frac{a}{2}+\sin.^2\frac{b}{2}-\sin.^2\frac{c}{2}}{2\sin.^2\frac{a}{2}\,.\,\sin.^2\frac{b}{2}}-2;$$

since

$$\cos.c-\cos.a\,.\,\cos.b=1-2\sin.^2\frac{c}{2}-\left(1-2\sin.^2\frac{a}{2}\right).\left(1-2\sin.^2\frac{b}{2}\right).$$

Also,
$$\cot.^2\frac{a}{2}+\cot.^2\frac{b}{2}+1=\frac{1-\sin.^2\frac{a}{2}}{\sin.^2\frac{a}{2}}\,.\,\frac{1-\sin.^2\frac{b}{2}}{\sin.^2\frac{b}{2}}+1$$

$$=\frac{1-\sin.^2\frac{a}{2}-\sin.^2\frac{b}{2}}{\sin.^2\frac{a}{2}\,.\,\sin.^2\frac{b}{2}}+2;$$

$$\therefore\ \frac{1}{\sin.^2\frac{S}{2}}=\frac{1-\sin.^2\frac{c}{2}}{\sin.^2\frac{a}{2}\,.\,\sin.^2\frac{b}{2}\,.\,\sin.^2C}=\frac{\cos.^2\frac{c}{2}}{\sin.^2\frac{a}{2}\,.\,\sin.^2\frac{b}{2}\,.\,\sin.^2C};$$

$$\therefore\ \sin.\frac{S}{2}=\frac{\sin.\frac{a}{2}\,.\,\sin.\frac{b}{2}\,.\,\sin.C}{\cos.\frac{c}{2}}$$

$$= \frac{\sqrt{\sin.(s) \,.\, \sin.(s-a) \,.\, \sin.(s-b) \,.\, \sin.(s-c)}}{2\cos.\frac{a}{2} \,.\, \cos.\frac{b}{2} \,.\, \cos.\frac{c}{2}},$$

the formula required.

45. Also, since $\cos.\frac{S}{2} = \cot.\frac{S}{2} \,.\, \sin.\frac{S}{2}$

$$\cos.\frac{S}{2} = \frac{1 + \cos.a + \cos.b + \cos.c}{4\cos.\frac{a}{2} \,.\, \cos.\frac{b}{2} \,.\, \cos.\frac{c}{2}}$$

$$= \frac{\cos.^2\frac{a}{2} + \cos.^2\frac{b}{2} + \cos.^2\frac{c}{2} - 1}{2\cos.\frac{a}{2} \,.\, \cos.\frac{b}{2} \,.\, \cos.\frac{c}{2}}.$$

46. Again, since $\frac{1 - \cos.A}{\sin.A} = \tan.\frac{A}{2}$;

$$\text{therefore, } \tan.\frac{S}{4} = \frac{1 - \cos.\frac{S}{2}}{\sin.\frac{S}{2}}$$

$$= \frac{1 - \cos.^2\frac{a}{2} - \cos.^2\frac{b}{2} - \cos.^2\frac{c}{2} + 2\cos.\frac{a}{2} \,.\, \cos.\frac{b}{2} \,.\, \cos.\frac{c}{2}}{\sqrt{\sin.s \,.\, \sin.(s-a) \,.\, \sin.(s-b) \,.\, \sin.(s-c)}}.$$

But the numerator may be put under the form

$$\left(1 - \cos.^2\frac{a}{2}\right) . \left(1 - \cos.^2\frac{b}{2}\right) - \left(\cos.\frac{a}{2} \,.\, \cos.\frac{b}{2} - \cos.\frac{c}{2}\right)^2$$

$$= \left(\sin.\frac{a}{2} \,.\, \sin.\frac{b}{2}\right)^2 - \left(\cos.\frac{a}{2} \,.\, \cos.\frac{b}{2} - \cos.\frac{c}{2}\right)^2$$

$$= \left(\sin.\frac{a}{2}.\sin.\frac{b}{2} + \cos.\frac{a}{2}.\cos.\frac{b}{2} - \cos.\frac{c}{2}\right).\left(\sin.\frac{a}{2}.\sin.\frac{b}{2} - \cos.\frac{a}{2}.\cos.\frac{b}{2} + \cos.\frac{c}{2}\right)$$

$$= \left(\cos.\frac{a-b}{2} - \cos.\frac{c}{2}\right) . \left(-\cos.\frac{a+b}{2} + \cos.\frac{c}{2}\right)$$

$$= 4 \sin.\left(\frac{s-b}{2}\right) \text{sn.} \left(\frac{s-a}{2}\right) \sin.\left(\frac{s}{2}\right) \sin.\left(\frac{s-c}{2}\right);$$

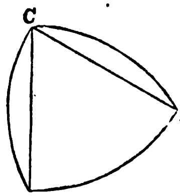
whence, since $\dfrac{\sin.\frac{a}{2}}{\sqrt{\sin.a}} = \sqrt{\dfrac{\sin.\frac{a}{2}}{2\cos.\frac{a}{2}}} = \dfrac{\sqrt{\tan.\frac{a}{2}}}{\sqrt{2}}$.

$$\therefore \tan.\frac{S}{4} = \sqrt{\tan.\frac{s}{2} \cdot \tan.\frac{s-a}{2} \cdot \tan.\frac{s-b}{2} \cdot \tan.\frac{s-c}{2}}.$$

This formula is due to Simon Lhuillier.

47. We proceed to give the investigation of certain theorems highly important in Geodetic operations, when a large extent of country is to be surveyed. The details are to be found in the article Trigonometry, in the Encyclopædia Metropolitana, in the Philosophical Transactions of the Royal Society, and also in the last edition of the Trigonometry of the late Professor Woodhouse.

48. Given two sides, and the included angle of a spherical triangle, to find the angle contained by the chords.



Let α, β, γ be the chords of CA, CB, and AB.

Let C be the included angle, and a, b, c, the sides of the triangle, and θ the angle required.

Now $\cos.c = \cos.a \cdot \cos.b + \sin.a \cdot \sin.b \cdot \cos.C$;

$$\text{or, } 1 - 2\sin.^2\frac{c}{2} = \left(1 - 2\sin.^2\frac{a}{2}\right) \cdot \left(1 - 2\sin.^2\frac{b}{2}\right)$$

$$+ 4\sin.\frac{a}{2} \cdot \sin.\frac{b}{2} \cdot \cos.\frac{a}{2} \cdot \cos.\frac{b}{2} \cdot \cos.C.$$

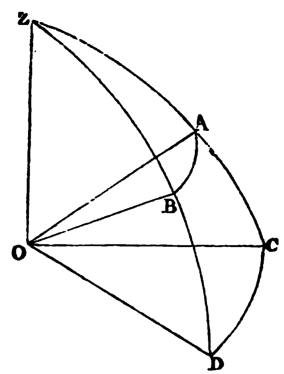
But $\gamma = 2\sin.\frac{c}{2}$, $\alpha = 2\sin.\frac{a}{2}$, and $\beta = 2\sin.\frac{b}{2}$;

$$\therefore\ \frac{-\gamma^2}{2} = -\frac{\alpha^2+\beta^2}{2} + \frac{\alpha^2\beta^2}{4} + \alpha\beta\,.\cos.\frac{a}{2}.\cos.\frac{b}{2}.\cos.C;$$

$$\therefore\ \frac{\alpha^2+\beta^2-\gamma^2}{2\alpha\beta} = \frac{\alpha\beta}{4} + \cos.\frac{a}{2}.\cos.\frac{b}{2}.\cos.C;$$

$$\text{or, } \cos.\theta = \sin.\frac{a}{2}.\sin.\frac{b}{2} + \cos.\frac{a}{2}.\cos.\frac{b}{2}.\cos.C.$$

49. Formula for the reduction of an oblique angle to the horizon.



Let A and B be the summits of two stations seen from O.

Let Z be the zenith, suppose two arcs ZBD, ZAC, to be drawn through Z and B, Z and A, to the horizon; then the $\angle\ AOB$ is the oblique, and the $\angle\ DOC$, or AZB, is the horizontal angle.

Let $AB = c$, $AC = H$, $BD = h$, $ZA = a$, $ZB = b$.

Then, in $\Delta\ AZB$ we have given, c, a, and b, to find AZB.

$$\text{Then } \because\ \cos.AZB = \frac{\cos.c - \cos.a\,.\cos.b}{\sin.a\,.\sin.b};$$

$$\therefore\ \sin.^2\frac{AZB}{2} = \frac{\sin.(s-a)\,.\sin.(s-b)}{\sin.a\,.\sin.b};$$

whence the angle may be found.

50. When H and h are small, and $\angle AZB$ does not differ much from AOB, to find the formula for the reduction.

Let $AZB = c + x$;

$$\therefore \ \cos.(c + x) = \frac{\cos.c - \sin.H \,.\, \sin.h}{\cos.H \,.\, \cos.h}.$$

But $\cos.H = 1 - \frac{H^2}{2}$, and $\cos.h = 1 - \frac{h^2}{2}$ nearly;

also, for $\sin.H$ and $\sin.h$, we may put H and h;

and $\cos.(c + x) = \cos.c \,.\, \cos.x - \sin.x \,.\, \sin.c$

$= \cos.c - x \,.\, \sin.c$ nearly;

therefore, $\cos.c - x \,.\, \sin.c = \dfrac{\cos.c - Hh}{1 - \dfrac{H^2 + h^2}{2}}$

$$= (\cos.c - Hh)\,.\left(1 + \frac{H^2 + h^2}{2}\right) \text{ nearly;}$$

$$\therefore \ x = \frac{Hh}{\sin.c} - \frac{\cos.c}{\sin.c} \,.\, \frac{H^2 + h^2}{2}.$$

Let $H + h = p$, $H - h = q$;

$$\therefore \ 4Hh = p^2 - q^2, \text{ and } 2(H^2 + h^2) = p^2 + q^2;$$

$$\therefore \ x = \tfrac{1}{4}\left(\frac{p^2 - q^2 - (p^2 + q^2) \,.\, \cos.c}{\sin.c}\right)$$

$$= \tfrac{1}{4}\left\{p^2 \,.\, \frac{1 - \cos.c}{\sin.c} - q^2 \,.\, \frac{1 + \cos.c}{\sin.c}\right\}$$

$$= \left(\frac{p}{2}\right)^2 . \tan.\frac{c}{2} - \left(\frac{q}{2}\right)^2 . \cot.\frac{c}{2};$$

which is the difference between the oblique and horizontal angle.

This theorem is due to Legendre, who makes use of it in the solution of the following problem.

"Given two sides of a spherical triangle, each nearly equal to 90°, to find the difference between the third side and the included angle."

For since H and h are very small, a and b are each nearly $= 90$; and therefore, the $\angle\ AZB$ is nearly measured by AB, and may be assumed $= c + x$, where x is very small.

51. Solution of spherical triangles, the sides of which are very small in comparison with the radius of the sphere.

Let (r) be the radius of the sphere.

Then, if a similar triangle be described upon a sphere of which the radius is unity, the sides of this triangle will be $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$.

$$\therefore\ \cos. A = \frac{\cos.\frac{a}{r} - \cos.\frac{b}{r}\,.\,\cos.\frac{c}{r}}{\sin.\frac{b}{r}\,.\,\sin.\frac{c}{r}}.$$

$$\text{But } \cos.\frac{a}{r} = 1 - \frac{a^2}{2r^2} + \frac{a^4}{24r^4} \text{ nearly;}$$

$$\text{and } \sin.\frac{b}{r} = \frac{b}{r} - \frac{b^3}{6r^3} \text{ nearly;}$$

whence, by substitution, the numerator becomes

$$\frac{b^2 + c^2 - a^2}{2r^2} + \frac{a^4 - b^4 - c^4}{24r^4} - \frac{b^2\,.\,c^2}{4r^4};$$

and the denominator becomes

$$\frac{bc}{r^2}\,.\left(1 - \frac{b^2 + c^2}{6r^2}\right);$$

$$\therefore \cos. A = \frac{b^2+c^2-a^2}{2bc} + \frac{a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2}{24bc.r^2}.$$

Let $A_{\prime}$ be the angle opposite the side (a), in the rectilineal triangle of which the sides are a, b, c.

Then, $\cos. A_{\prime} = \dfrac{b^2 + c^2 - a^2}{2bc}$;

and

$$4b^2c^2 \sin.^2 A_{\prime} = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4);$$

$$\therefore \cos. A = \cos. A_{\prime} - \frac{bc}{6r^2} . \sin.^2 A_{\prime}.$$

Let $A = A_{\prime} + x$; $\therefore \cos. A = \cos. A_{\prime} - x \sin. A_{\prime}$ nearly;

$$\therefore x \sin. A_{\prime} = \frac{bc . \sin.^2 A_{\prime}}{6r^2}; \quad \therefore x = \frac{bc . \sin. A_{\prime}}{6r^2}.$$

But $bc . \sin. A_{\prime} = 2$. area of the rectilineal triangle, the sides of which are a, b, c, and which does not sensibly differ from that of the spherical triangle;

$$\therefore x = \frac{\text{area}}{3r^2};$$

$$\therefore A_{\prime} = A - x = A - \frac{\text{area}}{3r^2}.$$

And similarly,

$$B_{\prime} = B - \frac{\text{area}}{3r^2}, \text{ and } C_{\prime} = C - \frac{\text{area}}{3r^2};$$

$$\therefore A_{\prime} + B_{\prime} + C_{\prime} = 180 = A + B + C - \frac{\text{area}}{r^2};$$

$$\therefore \frac{\text{area}}{r^2} = A + B + C - 180 = E, \text{ the spherical excess};$$

$$\therefore A_{\prime} = A - \frac{E}{3}, \; B_{\prime} = B - \frac{E}{3}, \; C_{\prime} = C - \frac{C}{3};$$

whence we obtain this rule;

In a spherical triangle whose sides are but little curved, if each of the angles be diminished by one-third of the spherical excess, and these angles be considered the angles of a plane triangle, the sides being the same as before, the solution of the plane triangle will give a sufficiently accurate result.

COR. Hence, since a plane may be considered to be a portion of a sphere, with an infinitely large radius, we may, by making r infinite, find the cosine of the angle of a plane triangle.

CHAPTER V.

REGULAR POLYHEDRONS. SOLIDITY OF PARALLELOPIPEDON.

52. To find the number of equal and regular polygons which can be described upon the surface of a sphere, so as exactly to cover it.

Let (N) be the number of faces,
(n) be the number of sides in each,
(m) be the number of angles at one point.

Now, area of polygon = the sum of the angles $-(n-2)\pi$;

$\therefore$ sum of the angles = area of polygon + $(n-2)\pi$.

But area of polygon $= \dfrac{\text{surface of sphere}}{N} = \dfrac{4\pi}{N}$, the radius being considered unity;

$$\therefore \text{ sum of angles} = \frac{4\pi}{N} + (n-2)\pi;$$

$$\therefore \text{ each angle} = \frac{4\pi}{nN} + \frac{(n-2)\pi}{n};$$

$$\therefore \text{ all the angles at one point} = \frac{4\pi m}{nN} + \frac{(n-2)m\pi}{n} = 2\pi;$$

whence we immediately find

$$N = \frac{4m}{2n-(n-2)m}.$$

And we must now inquire what integer values of (n) and (m) will give (N) also an integer value.

(1.) Let $n = 3$, or let the polygons be equilateral spherical triangles.

Then $N = \frac{4m}{6-m}$; and, since m cannot be less than (3),

$$\text{if } m = 3, \quad N = 4;$$
$$m = 4, \quad N = 8;$$
$$m = 5, \quad N = 20.$$

But, if $m = 6$, $N = \infty$; and, if (m) be greater than 6, the values of N become negative.

Hence the surface of a sphere may be covered with four, eight, or twenty, spherical and equilateral triangles.

(2.) Let $n = 4$, when $N = \frac{4m}{8 - 2m} = \frac{2m}{4 - m}$.

Then, if $m = 3$, $N = 6$;

but, if $m = 4$, $N = \infty$; and, if m be $>$ 4, N is negative;

$\therefore$ the number of equal four-sided spherical figures that will cover a sphere is six.

(3.) Let $n = 5$. Then, $N = \frac{4m}{10 - 3m}$.

If $m = 3$, $N = \frac{12}{10 - 9} = 12$;

$m = 4$, or any greater number, N is negative.

Hence the surface of a sphere may be covered by twelve pentagonal spherical polygons.

By substituting higher powers of (n) we shall find that the values of N, whatever may be (m), will become

negative; but we may show *à priori* that (n) must be less than six.

For $\because\ 2n-(n-2)m$ must have a positive sign;

$$\therefore\ \frac{2n}{n-2} \text{ must always } > m;$$

$$\therefore\ > 3, \ (3) \text{ being the least value of } m;$$

$$\therefore\ 2n > 3n-6;$$

$$\therefore\ 6 > n, \text{ or } (n) \text{ must be } < 6.$$

Hence, there can be only three regular spherical polygons which can be so combined as to cover a sphere.

Cor. 1. If chords be drawn subtending the arcs of these spherical polygons, regular solid figures will be formed, having equal plane faces; these solids are called *regular solids*.

Cor. 2. Hence we may have regular solids, having either equilateral triangles, or squares, or equilateral pentagons, for their faces.

Cor. 3. And hence we may form a solid of triangles,
(1.) Having 4 triangular faces, which is the Tetrahedron.
(2.) . . . . 8 . . . . Octahedron.
(3.) . . . 20 . . . . Icosahedron.

The first has three, the second four, and the third five angles at each solid angle.

We may also form a solid of squares; in which case we shall have a figure of six plane faces, and three angles at each solid angle; such a figure is the Cube, or Hexahedron. And, lastly, we may form a solid with twelve pentagonal faces, and three angles at each solid angle; this solid, which makes the fifth regular solid, is called the Dodecahedron.

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53. From what has preceded it will appear, that a sphere may be described about any regular solid, and that the centre of the sphere must lie in the line which is drawn perpendicular to any one of the plane faces, through the centre of the face; and hence that the centre will be the intersection of two such perpendiculars to two adjacent faces.

54. In any Polyhedron the number of solid angles, together with the number of plane faces, exceeds by two the number of edges.

Let A = number of solid angles;
E = number of edges.

But sum of all the angles $= nN = mA$;

$$\therefore\ A = \frac{Nn}{m}.$$

But, by Art. 52, we have

$$N \,.\, (2n-(n-2)\,.\,m) = 4m;\ \therefore\ m = \frac{2nN}{(n-2)\,.\,N+4};$$

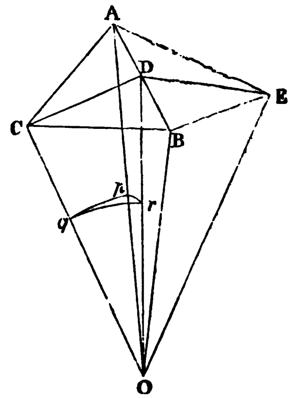
$$\therefore\ A = \frac{nN}{m} = \frac{(n-2)\,.\,N+4}{2} = \frac{nN}{2} - N + 2;$$

$$\therefore\ A + N = \frac{nN}{2} + 2.$$

But (Nn) is the number of edges before the polygons are joined together; and as two edges of the separated polygons make but one edge of the solid; $\therefore\ \frac{Nn}{2}$ = the number of edges of the solid $= E$;

$$\therefore\ A + N = E + 2.$$

55. To find the inclination of two adjacent faces of a regular polyhedron to each other.



Let AB be the common edge of two adjacent faces. C and E the centres of the faces. Draw CO, EO $\perp^r$ to the faces, and meeting in O; and CD, ED, $\perp^r$ to AB, the intersection of the planes ABC, ABE.

$\therefore \angle\ CDE$ is the angle of inclination required.

Join OD, and with centre O and radius $=$ unity, describe a spherical surface, meeting OC, OA, OD, respectively, in q, p, and r.

Let $n =$ number of sides in each face;
$m =$ plane angles in each solid angle.

Then, in the spherical triangle qpr, the $\angle\ qrp$ is a right angle, the sides qp, pr, qr, measure respectively the angles COA, AOD, DOC; and since the number of equal angles round (q) is the same as the number round C, and that $pqr = \angle\ ACD$;

$$\text{and } \because\ n \times ACB, \text{ or } 2n\,.\,ACD, = 2\pi;$$

$$\therefore\ ACD, \text{ or } pqr = \frac{\pi}{n}.$$

And $\because$ the number of angles which form the solid angle at A is (m), and that CA bisects one of the angles;

$$\therefore\ 2m \times qpr = 2\pi;\ \therefore\ qpr = \frac{\pi}{m}.$$

But, by Napier's Rules,

$$\cos. qpr = \cos. qr \times \sin. pqr;$$

$$\therefore\ \cos.(qr) = \cos. DOC = \sin. CDO = \frac{\cos. qpr}{\sin. pqr};$$

$$\text{or, } \sin.\left(\frac{CDE}{2}\right) = \frac{\cos.\left(\frac{\pi}{m}\right)}{\sin.\left(\frac{\pi}{n}\right)}.$$

56. OA and OC are evidently the radii of the circumscribing and inscribed spheres.

Let $OA = R$, and $OC = r$.

Now, $\cos. pq = \cot. qpr \times \cot. pqr$;

$$\text{or, } \cos. COA = \frac{OC}{OA} = \cot. \frac{\pi}{m} . \cot. \frac{\pi}{n};$$

$$\therefore \frac{R}{r} = \tan. \frac{\pi}{m} . \tan. \frac{\pi}{n}. \quad (1.)$$

Make $AD = a$; $\therefore AB = 2a$.

Then, $2a = 2CA . \sin. \frac{\pi}{n}$; $\therefore CA = \dfrac{a}{\sin. \frac{\pi}{n}}$.

And $OA^2 = OC^2 + CA^2$; $\therefore R^2 = r^2 + \dfrac{a^2}{\sin.^2 \frac{\pi}{n}}$. (2.)

And by means of (1) and (2) R and r may be determined.

57. Let the angle $CDE = C$, and $AB = 2a$,

$$\therefore r = CD . \tan. \frac{C}{2}; \quad CD = AD . \cot. \frac{\pi}{n};$$

$$\therefore r = a . \cot. \frac{\pi}{n} . \tan. \frac{C}{2};$$

$$R = r . \tan. \frac{\pi}{n} . \tan. \frac{\pi}{m} = a . \tan. \frac{\pi}{m} . \tan. \frac{C}{2}.$$

58. To find the solidity of the regular polyhedrons.

They may be divided into as many pyramids as there are faces, having a common vertex at O; and as the content of each pyramid is equal to the product of

the area of its base into the third of its altitude, therefore the content of the polyhedron = the number of faces × by the content of each pyramid is known.

Now,

$$\text{area of face} = n \times \Delta ACB = n \,.\, AD \,.\, CD = na^2 \cot. \frac{\pi}{n},$$

$$\text{altitude} = CO = r = a \cot. \frac{\pi}{n} \,.\, \tan. \frac{C}{2};$$

$$\therefore \text{ content of pyramid} = \frac{na^3}{3}\left(\cot. \frac{\pi}{2}\right)^2 \tan. \frac{C}{2};$$

$$\therefore \text{ polyhedron} = \frac{N \,.\, na^3}{3}\left(\cot. \frac{\pi}{n}\right)^2 \tan. \frac{C}{2};$$

59. To find $\frac{C}{2}$, r, R, and solidity in the five polyhedrons.

(1.) In the tetrahedron, $m = 3$, $n = 3$, and $N = 4$;

$$\frac{\pi}{m} = 60 = \frac{\pi}{n}; \quad \sin. \frac{C}{2} = \frac{\cos. 60}{\sin. 60} = \frac{1}{\sqrt{3}}; \quad \cos. C = \frac{1}{3};$$

$$\tan. \frac{C}{2} = \frac{1}{\sqrt{2}}; \quad \cot. \frac{\pi}{n} = \frac{1}{\sqrt{3}}; \quad \tan. \frac{\pi}{m} = \sqrt{3};$$

$$\therefore \ r = \frac{a}{\sqrt{6}}; \quad R = a\sqrt{\frac{3}{2}}; \quad \text{solidity} = \frac{a^3\sqrt{2}}{3}.$$

(2.) In the cube, $m = 3$, $n = 4$, $N = 6$;

$$\frac{\pi}{m} = 60; \quad \frac{\pi}{n} = 45; \quad \sin. \frac{C}{2} = \frac{\cos. 60}{\sin. 45} = \frac{1}{\sqrt{2}};$$

$$\therefore \ \frac{C}{2} = 45, \text{ and } C = 90;$$

$$\tan. \frac{C}{2} = 1; \quad \cot. \frac{\pi}{n} = 1; \quad \tan. \frac{\pi}{m} = \sqrt{3};$$

$$\therefore \ r = a, \ R = a\sqrt{3}; \quad \text{solidity} = 8a^3.$$

(3.) In the octahedron, $m = 4$, $n = 3$, $N = 8$;

$$\frac{\pi}{m} = 45; \quad \frac{\pi}{n} = 60; \quad \sin. \frac{C}{2} = \frac{\cos. 45}{\sin. 60} = \sqrt{\frac{2}{3}};$$

$$\therefore \cos. C = -\frac{1}{3}.$$

Hence the angles of inclination of the tetrahedron and octahedron are supplemental angles.

And $\tan. \frac{C}{2} = \sqrt{2}$; $\cot. \frac{\pi}{n} = \frac{1}{\sqrt{3}}$; $\tan. \frac{\pi}{m} = 1$;

$$\therefore r = a\sqrt{\frac{2}{3}};\ R = a\sqrt{2};\ \text{solidity} = \frac{8a^3\sqrt{2}}{3}.$$

(4.) In the dodecahedron, $m = 3$, $n = 5$, $N = 12$;

$$\frac{\pi}{m} = 60;\ \frac{\pi}{n} = 36;\ \sin. \frac{C}{2} = \frac{\cos. 60}{\sin. 36} = \frac{\sqrt{2}}{\sqrt{5 - \sqrt{5}}};$$

$$\tan. \frac{C}{2} = \frac{\sqrt{2}}{\sqrt{3 - \sqrt{5}}};\ \cot. \frac{\pi}{n} = \frac{1 + \sqrt{5}}{\sqrt{10 - 2\sqrt{5}}};\ \tan. \frac{\pi}{m} = \sqrt{3};$$

$$\therefore r = \frac{a}{2} \cdot \frac{1 + \sqrt{5}}{\sqrt{5 - 2\sqrt{5}}};\ R = a \cdot \frac{\sqrt{6}}{\sqrt{3 - \sqrt{5}}};$$

$$\text{solidity} = 8a^3 \cdot \frac{5 + 2\sqrt{5}}{\sqrt{6 - \sqrt{10}}}.$$

(5.) In the icosahedron, $m = 5$, $n = 3$, $N = 20$;

$$\frac{\pi}{m} = 36;\ \frac{\pi}{n} = 60;\ \sin. \frac{C}{2} = \frac{\cos. 36}{\sin. 60} = \frac{1 + \sqrt{5}}{2\sqrt{3}};$$

$$\therefore \cos. C = -\frac{\sqrt{5}}{3};\ \tan. \frac{C}{2} = \frac{1 + \sqrt{5}}{\sqrt{6 - 2\sqrt{5}}};$$

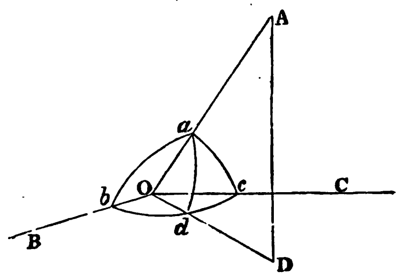
$$\cot. \frac{\pi}{n} = \frac{1}{\sqrt{3}};\ \tan. \frac{\pi}{m} = \frac{\sqrt{10 - 2\sqrt{5}}}{1 + \sqrt{5}};$$

$$\therefore r = a\frac{1 + \sqrt{5}}{\sqrt{18 - 6\sqrt{5}}} = a\frac{\sqrt{7 + 3\sqrt{5}}}{\sqrt{6}};$$

$$R = a\frac{\sqrt{5 - \sqrt{5}}}{\sqrt{3 - \sqrt{5}}} = \frac{a}{2}\sqrt{10 + 2\sqrt{5}};$$

$$\text{solidity} = \frac{10a^3}{3}\frac{\sqrt{2} + \sqrt{10}}{\sqrt{3 - \sqrt{5}}} = \frac{10}{3}a^3(3 + \sqrt{5}).$$

60. Given the three edges of a parallelopipedon, and the angles which they form together; find the solidity.



O, the point at which the edges meet. OA, one of the edges. OB, OC, the direction of the other two.

Let $OA = a$, and let (b) (c) be the other two edges.

Draw AD ⊥r to the base. Join OD, and with centre O, and radius unity, describe a spherical surface, cutting OA, OB, OC, OD, in a, b, c, d.

Let $\angle\ AOC = \alpha$, $\angle\ AOB = \beta$, $\angle\ BOC = \gamma$.

Then,

solidity $=$ area of base $\times AD = bc \,.\, \sin.\gamma \times a \,.\, \sin.AOD$.

But, from the right-angled spherical triangles,

$$\sin.ad = \sin.ab \times \sin.abd,$$

$$\text{or } \sin.AOD = \sin.\beta \,.\, \sin.abc.$$

But, if $s = \dfrac{\alpha + \beta + \gamma}{2} = \dfrac{ac + ab + bc}{2}$,

$$\sin.abc = \frac{2}{\sin.\beta \,.\, \sin.\gamma} \sqrt{\sin.(s) \,.\, \sin.(s-\alpha) \,.\, \sin.(s-\beta) \,.\, \sin.(s-\gamma)};$$

$$\therefore \text{ solidity} = abc \,.\, \sin.\gamma \,.\, \sin.\beta \,.\, \sin.(abc)$$

$$= 2abc\sqrt{\sin.(s) \,.\, \sin.(s-\alpha)\, \sin.(s-\beta) \,.\, \sin.(s-\gamma)}.$$

61. The same things being given, to find the diagonal of a parallelopipedon.

Referring to the same figure, let OD represent the diagonal of the base; but then the angle (adb) will not be a right angle.